Applications of Differentiation

Finding Extreme Values
Equations Between Functions
Inequalities Between functions
Limit Computations and l'Hospital's Rule

Finding the Extreme Values of Functions

Let f be a function which is continuous on the closed interval [*a*,*b*] and differentiable on the open interval (*a*,*b*).

To find the extreme values of f on the interval [a,b], perform the following steps:

1. Compute the derivative of the function f.
2. Find the zeros of the derivative in the interval (*a*,*b*).
3. Compute the values of f at the zeros of the derivative and at the end-points *a* and *b*.
4. Among these computed values, choose the largest and the smallest. These are the extreme values of the function f.

Finding Extreme Values

Example

Find the extreme values of the function $f(x) = x^3 - 2x + 2$ on the interval [0,2].

Solution

The function is differentiable everywhere. Hence the function will take its extreme values on the interval [0,2] either at points where the derivative vanishes or at the end-points of the interval.

Differentiation yields $f'(x) = 3x^2 - 2$. $f'(x) = 0 \Leftrightarrow x = \pm\sqrt{2/3} = \pm\sqrt{6}/3$.

Computing values we observe that $f(0) = 2$, $f(2) = 6$ and

$f\left(\frac{\sqrt{6}}{3}\right) \approx 0.911$. Hence f attains its maximum value on [0,2]

at $x = 2$ and minimum at $x = \sqrt{6}/3$.

Functional Equalities (1)

Examples

1 Show that $\cos^2 x + \sin^2 x = 1.$

Calculus is not needed to prove this well known formula – rather the derivation of several differentiation formulae need this trigonometric formula. It is, however, instructive to see how this follows using the methods of calculus.

Proof Consider the function $f(x) = \cos^2 x + \sin^2 x.$

Straight forward differentiation gives

$$f'(x) = 2\cos(x)(-\sin(x)) + 2\sin(x)\cos(x) = 0.$$

Hence f is a constant function. The equality of the example follows from the fact that $f(0) = 1.$ ■

Functional Equalities (2)

Examples

2 Show that $\arcsin\left(\frac{x-1}{x+1}\right) + \frac{\pi}{2} = 2\arctan\left(\sqrt{x}\right)$ for $x \geq 0.$

Proof Consider the function

$$f(x) = \arcsin\left(\frac{x-1}{x+1}\right) + \frac{\pi}{2} - 2\arctan\left(\sqrt{x}\right).$$

Straight forward differentiation gives $f'(x) = 0.$

Use a computer algebra system here!

Hence f is a constant function. The equality of the example follows from the fact that $f(0) = 0.$ ■

Estimating Functions (1)

Derivates provide a powerful tool to study the behavior of functions both locally – near a given point – and globally. Limit computations are examples of the former applications. Latter applications often have to do with functional estimates.

Examples

1 Show that $\sin(x) \leq x$ for $x \geq 0.$

Proof The function $f(x) = x - \sin(x)$ is increasing for $x \geq 0$ since $f'(x) = 1 - \cos(x) \geq 0$ with equality only if $x = n2\pi,\ n \in \mathbb{Z}.$

It follows that $f(x) \geq f(0) = 0$ for $x \geq 0.$ This proves the claim. ■

Estimating Functions (2)

2 Show that $\frac{x}{2} \leq \sin(x) \leq x$ for $0 \leq x \leq 1$.

Proof By Problem 1 we already know that $\sin(x) \leq x$ for $0 \leq x$.

It remains to show that $\frac{x}{2} \leq \sin(x)$ for $0 \leq x \leq 1$.

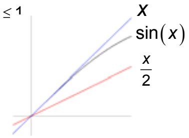
Consider the function $f(x) = \sin(x) - \frac{x}{2}$.

Clearly $f(0)=0$ and $f'(x)=\cos(x) - \frac{1}{2} > 0$ for $0 \leq x \leq 1$

We conclude that f is increasing for $0 \leq x \leq 1$.

Hence, for $0 \leq x \leq 1$, $f(x) \geq f(0) = 0$.

This implies: $\sin(x) \geq \frac{x}{2}$ for $0 \leq x \leq 1$. ■



Estimating Functions (3)

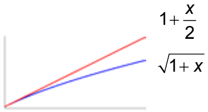
3 Show that $\sqrt{1+x} \leq 1 + \frac{x}{2}$ for $x \geq 0$.

Proof The function $f(x) = 1 + \frac{x}{2} - \sqrt{1+x}$ is increasing for $x \geq 0$

since $f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}} \geq 0$ with equality only if $x = 0$.

It follows that $f(x) \geq f(0) = 0$ for $x \geq 0$.

This proves the claim. ■

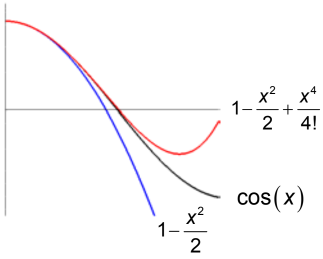


Estimating Functions (4)

4 Show that $1 - \frac{x^2}{2} \leq \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{4!}$ for $x \geq 0$.

The figure illustrates this double inequality. Plotting functions can often be used in an experimental way to try to find estimates.

Mathematical proves cannot, however, be based on looking at graphs only.



Estimating Functions (5)

5 Show that $1-\frac{x^2}{2}\le\cos(x)\le 1-\frac{x^2}{2}+\frac{x^4}{4!}$ for $x\ge 0$.

Proof We prove that $\cos(x)\le 1-\frac{x^2}{2}+\frac{x^4}{4!}$ for $x\ge 0$.

The other estimate follows in the same way.

Consider the function $f(x)=1-\frac{x^2}{2}+\frac{x^4}{4!}-\cos(x)$.

$$f'(x)=-x+\frac{x^3}{3!}+\sin(x),\ f''(x)=-1+\frac{x^2}{2!}+\cos(x),$$

$$f'''(x)=x-\sin(x).$$

$\sin(x)\le x\Rightarrow f''(x)$ increasing. Since $f''(0)=0$, $f''(x)>0$.

Hence f' is increasing. Since $f'(0)=0$, $f'(x)>0$.

Hence f is increasing. Since $f(0)=0$, $f(x)>0$. ■

Evaluating Indeterminates (1)

When trying to evaluate expressions for certain values of the parameters of variables, the result may be of the form

$$\frac{0}{0},\ 0\times\infty,\ \frac{\infty}{\infty},\ \infty-\infty,\ \text{or}\ 0^0.$$

These are called indeterminate forms, or NaN's (Not a Number).

Using differentiation, it is often possible to analyze the behavior of the expression in question, and then determine the actual value of the indeterminate form as a certain limit.

Example Determine $\lim_{x\to 0}\frac{\cos(x)-1}{x^2}$.

Solution follows

Evaluating Indeterminates (2)

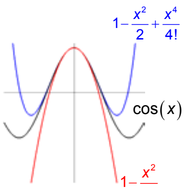
Example Determine $\lim_{x\to 0}\frac{\cos(x)-1}{x^2}$.

Solution

Use the estimate $1-\frac{x^2}{2}\le\cos(x)\le 1-\frac{x^2}{2}+\frac{x^4}{4!}$

which is actually valid for all x to get

$$-\frac{1}{2}\le\frac{\cos(x)-1}{x^2}\le-\frac{1}{2}+\frac{x^2}{4!}.$$



Hence $\lim_{x\to 0}\left(-\frac{1}{2}\right)=-\frac{1}{2}\le\ \lim_{x\to 0}\frac{\cos(x)-1}{x^2}\le\lim_{x\to 0}\left(-\frac{1}{2}+\frac{x^2}{4!}\right)=-\frac{1}{2}$.

Conclude $\lim_{x\to 0}\frac{\cos(x)-1}{x^2}=-\frac{1}{2}$.

L’Hospital’s Rule (1)

L’Hospital’s Rule

Assume that the functions f and g are differentiable and that $g'(x) \neq 0$ near *a*. Suppose that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \quad \text{provided that the latter limit exists.}$$

The formula holds even if the limit on the right limit is $\pm\infty$.

Remark L’Hospital’s Rule can also be applied to one-sided limits or to limits at the positive or negative infinity.

L’Hospital’s Rule (2)

Proof of a special case of l’Hospital’s Rule

Assume that f' and g' are continuous at a finite number *a* and that $g'(a) \neq 0$. Suppose that $f(a) = g(a) = 0$.

$$\text{Then } \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

In this rewriting we use the fact that f(*a*)=g(*a*)=0.

Proof

$$\frac{f(x)}{g(x)} = \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} \xrightarrow[x\to a]{} \frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

In the above we assume that $x \neq a$. The continuity of the derivatives f' and g' imply the last equation. ■

L’Hospital’s Rule (3)

Example Determine $\lim_{x\to 0}\frac{\ln(1+x)}{x}$.

Solution

Apply l'Hospital's Rule to get

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = \lim_{x\to 0}\frac{1}{1+x} = 1.$$

L'Hospital's Rule (4)

Example

Determine $\lim\limits_{x\to\infty}\frac{\ln(x)}{x}.$

Solution

Apply l'Hospital's Rule to get

$$\lim_{x\to\infty}\frac{\ln(x)}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{x}=0.$$

L'Hospital's Rule (5)

Example

Determine $\lim\limits_{x\to\infty}\frac{x^2}{2^x}.$

Solution

Apply l'Hospital's Rule twice to get

$$\lim_{x\to\infty}\frac{x^2}{2^x}=\lim_{x\to\infty}\frac{2x}{2^x\ln 2}=\lim_{x\to\infty}\frac{2}{2^x(\ln 2)^2}=0.$$

L'Hospital's Rule (6)

Example

Determine $\lim\limits_{x\to 0+}x\ln x.$

Solution

Apply l'Hospital's Rule to get

$$\lim_{x\to 0+}x\ln x=\lim_{x\to 0+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0+}(-x)=0.$$

This rewriting allowed us to use l'Hospital's Rule.

L’Hospital’s Rule (7)

Example

Determine $\lim_{x\to 0+} x^x$.

Solution

$$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{x\ln x} = e^{\lim_{x\to 0+} x\ln x} = e^0 = 1.$$

Here we use the result of the previous example and the fact that the exponential function is continuous.