
Differential Equations

Ordinary Differential Equations
Exponential Growth or Decay
Families of Solutions
Separation of Variables
Numerical Solving – Direction Fields
Hybrid Numerical-Symbolic Solving
Orthogonal Curves
Modeling

Ordinary Differential Equations

Definition

A **differential equation** is an equation involving derivatives of an unknown function and possibly the function itself as well as the independent variable.

Example

$$y' = \sin(x), \quad (y')^4 - y^2 + 2xy - x^2 = 0, \quad y'' + y^3 + x = 0$$

1st order equations

2nd order equation

Definition

The **order** of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation

In the simplest cases, equations may be solved by direct integration.

Examples

$$y' = \sin(x) \Rightarrow y = -\cos(x) + C$$

$$y'' = 6x + e^x \Rightarrow y' = 3x^2 + e^x + C_1 \Rightarrow y = x^3 + e^x + C_1x + C_2$$

Observe that the set of solutions to the above 1st order equation has 1 parameter, while the solutions to the above 2nd order equation depend on two parameters.

Exponential Growth or Decay

Definition

The **differential equation** $y' = ky$ is an equation of exponential growth if $k > 0$ and of exponential decay if $k < 0$.

Remark

In the above equation, the name of the variable is not indicated. We may choose that freely. Usually this type of a differential equation models a development over time like growth of a deposit in a bank or a population growth. Hence we may want to call the variable t .

Solution

$$y' = ky \Rightarrow \frac{y'(t)}{y(t)} = k \Rightarrow \int \frac{y'(t)}{y(t)} dt = \int k dt$$

Now substitute $y = y(t)$.

$$\Rightarrow \int \frac{dy}{y} = \int k dt \Rightarrow \ln|y| = kt + C_1 \Rightarrow y = \pm e^{C_1} e^{kt}$$

This computation is valid assuming that $y \neq 0$.

Writing $C = \pm e^{C_1}$ we get $y = C e^{kt}$, $C \neq 0$.

Direct computation shows that $y = 0$ is a special solution of the above equation.

General solution $y = C e^{kt}$, $C \in \mathbb{R}$.

Families of Solutions

Example

$$9yy' + 4x = 0$$

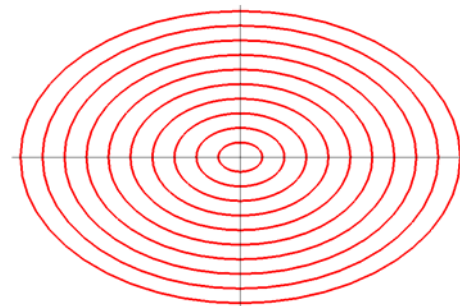
Solution

$$\int (9yy' + 4x) dx = C_1 \Rightarrow \int 9y(x)y'(x) dx + \int 4x dx = C_1$$

$$\Rightarrow \int 9y dy + 2x^2 = C_1 \Rightarrow \frac{9y^2}{2} + 2x^2 = C_1 \Rightarrow 9y^2 + 4x^2 = 2C_1$$

This yields $\frac{y^2}{4} + \frac{x^2}{9} = C$ where $C = \frac{C_1}{18}$.

Observe that given any point (x_0, y_0) , there is a unique solution curve of the above equation which goes through the given point.



The solution is a family of ellipses.

Separation of Variables

Definition

A differential equation of the type $y' = f(x)g(y)$ is **separable**.

Example

$y' = xy$ and $y' = \frac{x}{y}$ are separable

but $y' = \frac{x+y}{x-y}$ is not separable.

Separable differential equations can often be solved with direct integration. This may lead to an equation which defines the solution implicitly rather than directly.

Example

$$y' = \frac{x}{y} \Rightarrow y(x)y'(x) = x \Rightarrow \int y(x)y'(x)dx = \int xdx$$

$$\Rightarrow \int ydy = \int xdx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C_1 \Rightarrow y^2 = x^2 + C$$

Substitute $y = y(x)$
to simplify this
integral.

Separation of Variables

Example

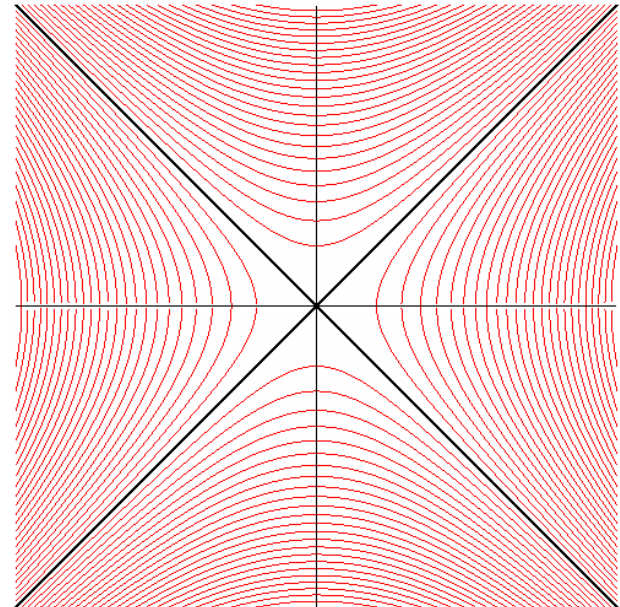
$$y' = \frac{x}{y} \Rightarrow y(x)y'(x) = x$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C_1 \Rightarrow y^2 = x^2 + C$$

The picture on the right shows some solutions to the above differential equation. The straight lines

$$y = x \text{ and } y = -x$$

are special solutions. A unique solution curve goes through any point of the plane different from the origin. The special solutions $y = x$ and $y = -x$ go both through the origin.



General Separable Equations

Consider the separable equation $y' = f(x)g(y)$.

Observe that if y_0 is such that $g(y_0)=0$, then the constant function $y = y_0$ is a solution to the above differential equation. Hence all solutions to the equation $g(y) = 0$ give special solutions to the above differential equation.

$$y'(x) = f(x)g(y(x)) \Rightarrow \frac{y'(x)}{g(y(x))} = f(x)$$

This computation is valid provided that $g(y) \neq 0$. We get:

$$\int \frac{y'(x)}{g(y(x))} dx = \int f(x) dx$$

Substitute $y = y(x)$.

$$\int \frac{dy}{g(y)} = \int \frac{dx}{f(x)}$$

If integration can be performed, this usually leads to an equation that defines y implicitly as a function of x .

Separation of Variables

Example

$$y' = \frac{x^2 + x}{y^2 + y}$$

This notation is due to Leibniz.

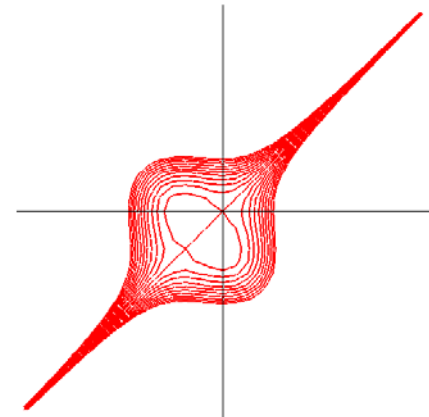
Solution

$$y' = \frac{x^2 + x}{y^2 + y} \Rightarrow \frac{dy}{dx} = \frac{x^2 + x}{y^2 + y} \Rightarrow (y^2 + y)dy = (x^2 + x)dx$$

$$\Rightarrow \frac{y^3}{3} + \frac{y^2}{2} = \frac{x^3}{3} + \frac{x^2}{2} + C_1 \quad \Rightarrow 2y^3 + 3y^2 = 2x^3 + 3x^2 + 6C_1$$

$$\Rightarrow 2y^3 + 3y^2 = 2x^3 + 3x^2 + C, \text{ where } C = 6C_1.$$

It is, in principle, possible to solve y in terms of x and C from the above implicit solution. This would lead to very long expressions. The picture on the right shows some solution curves.



Numerical Solving

Solutions to a differential equation of the type $y' = f(x,y)$ can always be approximated numerically by computing the **direction field** or the **slope field** defined by this equation.

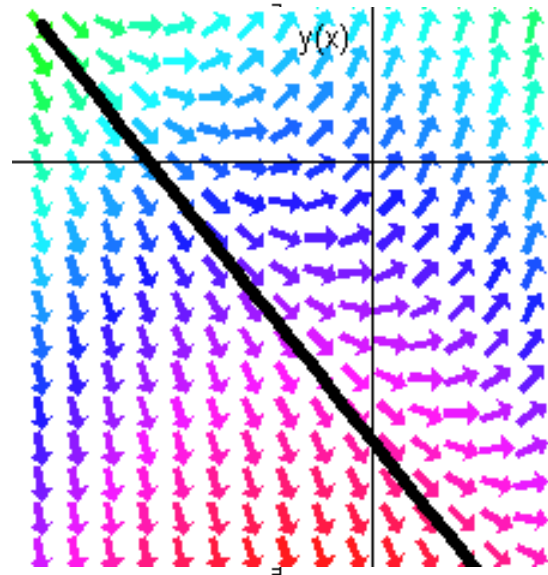
The computation has the following steps:

1. Choose first a rectangle in the xy -plane in which rectangle you want to approximate solutions.
2. Form a grid of points of the rectangle. Choose the points so that they cover the rectangle in question evenly.
3. At each grid point (x,y) compute the value of the function $f(x,y)$.
4. Starting from each grid point draw a short arrow with slope $f(x,y)$.
5. Connect arrows to form an approximation of a solution curve.

Numerical Solving

The picture on the right shows the direction field of the differential equation $y' = x + y + 1$.

Connecting arrows, one can approximate solutions. There is one special solution. Can you find it?



The special solution is $y = -x - 2$. This can easily be verified by a direct computation.

Hybrid Numerical-Symbolic Solving

By plotting the direction field of the differential equation

$$y' = x + y + 1$$

we found the special solution $y = -x - 2$.

To find the general solution, substitute $y = -x - 2 + v$ to the original equation and solve for v (which is a new unknown function).

One gets $y' = -1 + v'$ and the equation for v is

$$-1 + v' = x + (-x - 2 + v) + 1.$$

This simplifies to $v' = v$ which can be solved by direct integration.

$$\frac{dv}{dx} = v \Rightarrow \frac{dv}{v} = dx \Rightarrow \int \frac{dv}{v} = \int dx \quad \ln|v| = x + C_1 \Rightarrow v = \pm e^{C_1} e^x = C e^x.$$

Conclusion

The general solution is $y = -x - 2 + C e^x$.

Orthogonal Curves (1)

Example

Consider the family of parabola $y = x^2 + C$. Find the family of curves which intersect the above family of parabola perpendicularly.

Solution

By differentiation we get: $y' = 2x$. Hence the family of parabola in question satisfies the differential equation $y' = 2x$.

Two curve intersect perpendicularly if the product of the slopes of the tangents at the intersection point is -1 . This gives the following differential equation for the orthogonal family of curves.

$$y' = -\frac{1}{2x}$$

Orthogonal Curves (2)

Example

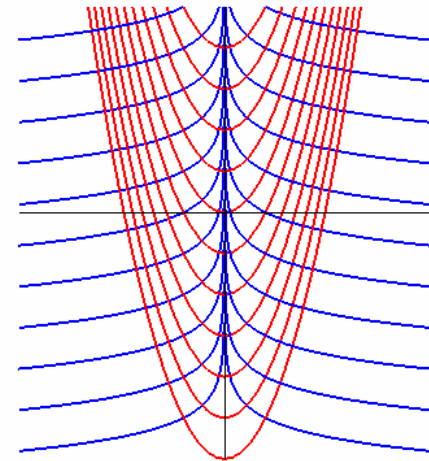
Consider the family of parabola $y = x^2 + C$. Find the family of curves which intersect the above family of parabola perpendicularly.

Solution (cont'd)

It remains to solve $y' = -\frac{1}{2x}$.

$$y' = -\frac{1}{2x} \Rightarrow \int dy = -\frac{1}{2} \int \frac{1}{x} dx \Rightarrow y = -\frac{1}{2} \ln|x| + C$$

The figure on the right shows these two orthogonal families of curves.



Newton's Law of Heating and Cooling

Newton's Law

The temperature of a hot or a cold object decreases or increases at a rate proportional to the difference of the temperature of the object and that of its surrounding.

Let $H(t)$ be the temperature of the hot or cold object at time t . Let H_∞ be the temperature of the surrounding. With this notation Newton's Law can be expressed as the differential equation

$$H'(t) = k (H(t) - H_\infty).$$

In Leibniz's notation:

$$\frac{dH}{dt} = k (H - H_\infty).$$

The unknown coefficient k must be determined experimentally.

Cooking a Turkey

Example

A turkey is put in a oven heated to 300 degrees (F). Initially the temperature of the turkey is 70 degrees. After one hour the temperature is 86 degrees. How long does it take until the temperature of the turkey is 180 degrees?

Solution

The differential equation is $H' = k(H - 300)$.

$$\frac{dH}{dt} = k(H - 300) \Rightarrow \frac{dH}{H - 300} = k dt$$

$$\Rightarrow \int \frac{dH}{H - 300} = \int k dt \Rightarrow \ln|H - 300| = kt + C_1$$

$$\Rightarrow H - 300 = \pm e^{C_1} e^{kt} \Rightarrow H = 300 + C e^{kt}.$$

Here the coefficient C may take any value including 0.

The unknown quantities k and C need to be determined from the information ($H(0) = 70$ and $H(1) = 86$) given in the problem.

Cooking a Turkey

Example

A turkey is put in a oven heated to 300 degrees (F). Initially the temperature of the turkey is 70 degrees. After one hour the temperature is 100 degrees. How long does it take until the temperature of the turkey is 180 degrees?

Solution (cont'd)

We now know that $H = 300 + C e^{kt}$.
 $H(0) = 70 \Rightarrow 70 = 300 + C e^0 \Rightarrow C = -230$.

Hence the model is $H = 300 - 230 e^{kt}$. To determine k use the fact that after one hour the temperature is 100 degrees.

$$H(1) = 100 \Rightarrow 100 = 300 - 230 e^k \quad e^k = \frac{200}{230} \Rightarrow k = \ln\left(\frac{200}{230}\right) \approx -0.1398$$

Model for the Temperature of the Turkey

$$H = 300 - 230 e^{-0.1398t}$$

Solving t from the equation $H(t) = 180$ we get $t = 4.65$. Hence the turkey is done after about 4 hours and 40 minutes.

Medical Modeling

Example

Half-life of morphine in the body is 2 hours. At time $t = 0$ a patient is given a dose of 5 mg of morphine. How much morphine is left after 3 hours? Assume that the rate at which morphine is eliminated is proportional to the amount of morphine left.

Solution

Let $m(t)$ = the amount of morphine at time t .

By the assumptions we have $\frac{dm}{dt} = km$. This can be solved by direct integration.

The model is $m = Ce^{kt}$.

$m(0) = 5 \Rightarrow 5 = Ce^0 \Rightarrow C = 5$. The model is now $m(t) = 5e^{kt}$.

$m(2) = 2.5 \Rightarrow 2.5 = 5e^{k \times 2} \Rightarrow 2k = \ln\left|\frac{1}{2}\right| \Rightarrow k = -\frac{\ln 2}{2} \approx -0.347$.

Model for the Amount of Morphine at Time t $m(t) = 5e^{-0.347t}$.

We get $m(3) \approx 1.77$ mg.