

# Mean Value Theorem

Increasing Functions  
Extreme Values of Functions  
Rolle's Theorem  
Mean Value Theorem

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## Increasing Functions (1)

Assume that the function  $f$  is everywhere increasing and differentiable.

Then  $\forall h \neq 0 : \frac{f(x+h) - f(x)}{h} \geq 0$ . It follows that  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \geq 0$ .

This follows from the fact that  $f$  is increasing.

Differentiability implies that the limit exists.

### Theorem

Assume that the function  $f$  is differentiable at a point  $x_0$  with  $f'(x_0) > 0$ . Then  $\exists \delta > 0$  such that

- 1)  $f(x) > f(x_0)$  for  $x_0 < x < x_0 + \delta$ , and
- 2)  $f(x) < f(x_0)$  for  $x_0 - \delta < x < x_0$ .

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## Increasing Functions (2)

### Theorem

Assume  $f'(x_0) > 0$ .  $\exists \delta > 0$  such that

- 1)  $f(x) > f(x_0)$  for  $x_0 < x < x_0 + \delta$ , and
- 2)  $f(x) < f(x_0)$  for  $x_0 - \delta < x < x_0$ .

### Proof

By the definition of the derivative and using the fact that  $f'(x_0) > 0$ :  $\exists \delta > 0$  such that

$$0 < |x - x_0| < \delta \Rightarrow \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < f'(x_0).$$

$$\text{Hence } 0 < |x - x_0| < \delta \Rightarrow \frac{f(x) - f(x_0)}{x - x_0} > 0.$$

This implies that if  $x - x_0$  is positive, then also  $f(x) - f(x_0)$  is positive proving the first statement. The second statement follows similarly. ■

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## Local Extreme Values

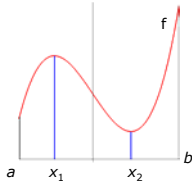
### Definition

A point  $x_1$  in the open interval  $(a,b)$  is a **local maximum** of the function  $f$  if, for values of  $x$  near the point  $p$ ,  $f(x) \leq f(x_1)$ .

A point  $x_2$  in the open interval  $(a,b)$  is a **local minimum** of the function  $f$  if, for values of  $x$  near the point  $p$ ,  $f(x) \geq f(x_2)$ .

By a local **extreme point** of a function we mean either a local maximum or a local minimum. Values of a function at its extreme points are called **extreme values**.

In the figure on the right, the point  $x_1$  is a local maximum and the point  $x_2$  a local minimum of the function  $f$ .



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## Criteria for Local Extreme Values

### Theorem

Let  $f$  be a function which is continuous on the closed interval  $[a,b]$  and differentiable on the open interval  $(a,b)$ .

If  $c \in (a,b)$  is a local maximum or a local minimum of the function  $f$ , then the derivative of  $f$  vanishes at the point  $c$ , i.e.  $f'(c)=0$ .

### Proof

Assume that  $c \in (a,b)$  is a local maximum or a local minimum of the function  $f$ .

If  $f'(c) > 0$ , then, by [the previous Theorem](#), near the number  $c$  and on the right hand side of  $c$ , the function  $f$  takes values that are larger than  $f(c)$ . On the left hand side of the number  $c$  the function takes values which are smaller than  $f(c)$ . Hence  $c$  cannot be a local extreme value.

In the same way we see that  $f'(c)$  cannot be negative. We conclude that  $f'(c) = 0$ . ■

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## Extreme Values of Continuous Functions

### Theorem

A function  $f$  which is continuous on a closed interval  $[a,b]$  takes its maximum and minimum values on  $[a,b]$ .

We will not prove the result here. By geometric arguments the result appears plausible, and a rigorous proof uses arguments related to those used in the proof of the [Intermediate Value Theorem](#) for Continuous Functions.

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## Finding the Extreme Values

Let  $f$  be a function which is continuous on the closed interval  $[a,b]$  and differentiable on the open interval  $(a,b)$ .

To find the extreme values of  $f$  on the interval  $[a,b]$ , perform the following steps:

1. Compute the derivative of the function  $f$ .
2. Find the zeros of the derivative in the interval  $(a,b)$ .
3. Compute the values of  $f$  at the zeros of the derivative and at the end-points  $a$  and  $b$ .
4. Among these computed values, choose the largest and the smallest. These are the extreme values of the function  $f$ .

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## Rolle's Theorem

**Theorem** Let  $f$  be a function which satisfies the following conditions

1.  $f$  is continuous on the closed interval  $[a,b]$ ,
2.  $f$  is differentiable on the open interval  $(a,b)$ , and
3.  $f(a)=f(b)$ .

Then there is a point  $c \in (a,b)$  such that the derivative of  $f$  vanishes at the point  $c$ , i.e.,  $f'(c) = 0$ .

**Proof** If  $f(x)=f(a)=f(b)$  for all  $x$  between  $a$  and  $b$ , then  $f$  is a constant function, and the derivative of  $f$  vanishes for all  $x$ , and we can take  $c$  to be any point between  $a$  and  $b$ .

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## Rolle's Theorem

**Theorem** Assume that  $f$  is continuous on  $[a,b]$  and differentiable on  $(a,b)$ , and  $f(a)=f(b)$ . Then there is a point  $c \in (a,b)$  such that  $f'(c) = 0$ .

**Proof (cont'd)**

If  $f$  is not a constant function, then either its maximum on  $[a,b]$  is greater than  $f(a)$  or its minimum is smaller than  $f(a)$ .

Assume that the maximum of  $f$  on  $[a,b]$  is greater than  $f(a)$ . Then, since  $f(b) = f(a)$ ,  $f$  takes its maximum at a point  $c \in (a,b)$ .

By the [Criteria for Local Extreme Values](#), we conclude that  $f'(c) = 0$ .

If the maximum of  $f$  on  $[a,b]$  is not greater than  $f(a)$ , then the minimum is smaller. And one can apply the Criteria for Local Extreme Values to the minimum of  $f$  to conclude the existence of  $c$ . ■

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## The Mean Value Theorem

**Theorem** Let  $f$  be a function which satisfies the following conditions

1.  $f$  is continuous on the closed interval  $[a, b]$ , and
2.  $f$  is differentiable on the open interval  $(a, b)$ .

Then there is a point  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This condition can be expressed equivalently as

$$f(b) - f(a) = f'(c)(b - a).$$

**Proof** Apply Rolle's Theorem to the function

$$g(x) = f(x) + (x - a) \frac{f(a) - f(b)}{b - a} \quad \blacksquare$$

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## Corollary of the Mean Value Theorem

**Theorem**

Assume that the functions  $f$  and  $g$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

If  $f'(x) = g'(x)$  for all  $x \in (a, b)$ , then

$f - g$  is a constant function.

**Proof** Apply the Mean Value Theorem to the function  $h = f - g$ .

$\forall x_1$  and  $x_2$ ,  $a \leq x_1 < x_2 \leq b$ ,  $\exists c \in (x_1, x_2)$  such that

$$h(x_2) - h(x_1) = h'(c)(x_2 - x_1).$$

Since  $h'(c) = 0$  by the assumptions and by the definition of the function  $h$ ,  $h(x_2) = h(x_1)$ , i.e.,  $h$  is a constant function.  $\blacksquare$

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## Increasing Functions

**Theorem**

Assume that the function  $f$  is everywhere differentiable on an open interval  $(a, b)$ , and that  $\forall x \in (a, b): f'(x) > 0$ .

Then  $f$  is increasing on  $(a, b)$ .

**Proof** Let  $x_1 < x_2$ . We have to show that  $f(x_1) < f(x_2)$ .

By the Mean Value Theorem  $c \in (x_1, x_2)$  such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

By the assumptions,  $f'(c) > 0$ . Hence also  $f'(c)(x_2 - x_1) > 0$ , i.e.  $f(x_2) - f(x_1) > 0$ . This means that  $f(x_1) < f(x_2)$ .  $\blacksquare$

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## Increasing Functions (2)

### Remark

The condition " $\forall x \in (a, b): f'(x) > 0$ " of the previous theorem can be slightly relaxed: it is enough that  $f'(x) > 0$  for all  $x$  except for a finite number of values of  $x$ .

### Example

The function  $f(x) = x^3$  is increasing even though  $f'(x) = 3x^2 = 0$  for  $x = 0$ .



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## Decreasing Functions

### Theorem

Assume that  $f$  is everywhere differentiable on  $(a, b)$  and that  $f'(x) < 0$  for all  $x$  possibly excluding a finite number of values of  $x$ . Then the function  $f$  is decreasing on  $(a, b)$ .

### Proof

Let  $g = -f$ . Then the function  $f$  satisfies the conditions of the previous theorem, and  $g$  is increasing. Hence  $f$  is decreasing. ■

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