Intermediate Value Theorem for Continuous Functions

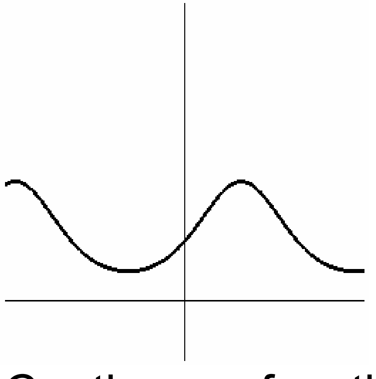
Review of Properties of Continuous Functions
Function that is Continuous at Irrational Points Only
Bolzano’s Theorem
Intermediate Value Theorem for Continuous Functions
Applications of the Intermediate Value Theorem

Continuous Functions (1)

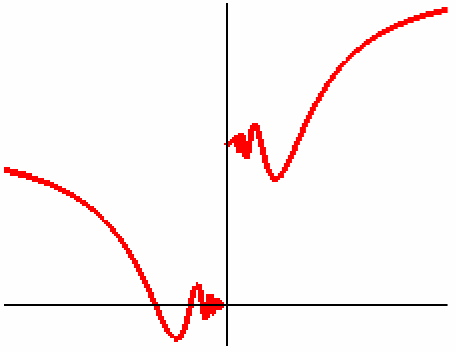
Definition 1 A function $f:\mathbb{R}\to\mathbb{R}$ is continuous at $x = x_0$ if the limit $\lim_{x\to x_0} f(x) = f(x_0)$.

The function $f:\mathbb{R}\to\mathbb{R}$ is continuous in an interval if it is continous at each point of the interval.

A function which is not continuous (at a point or in an interval) is said to be discontinuous.



Continuous function



Discontinuous function

Continuous Functions (2)

Definition 2 A function $f:\mathbb{R}\to\mathbb{R}$ is continuous at $x = x_0$ if

$$\forall\varepsilon>0:\exists\delta>0 \text{ such that } |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon.$$

Lemma Assume that $f:\mathbb{R}\to\mathbb{R}$ is continuous at $x = x_0$ and that $f(x_0)>0$. Then $\exists\delta>0$ such that $|x-x_0|<\delta \Rightarrow f(x)>0$.

Proof Choose $\varepsilon = f(x_0)$ in the definition of continuity.

We can do this since, by the assumption, $f(x_0)>0$.

Hence

$$\exists\delta>0 \text{ such that } |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<f(x_0).$$

$$|f(x)-f(x_0)|<f(x_0) \Leftrightarrow f(x)-f(x_0)<f(x_0) \wedge f(x)-f(x_0)>-f(x_0)$$

$$\Leftrightarrow f(x)<2f(x_0) \wedge f(x)>0.$$

Continuous Functions (3)

The following functions are continuous at all points where they take finite values.

1. Polynomials – they are continuous everywhere
2. Rational functions
3. Functions defined by algebraic expressions
4. Exponential functions and their inverses
5. Trigonometric functions and their inverses

Continuous Functions (4)

Assume that f and g are continuous functions.

Theorem The following functions are continuous:

1. $f + g$
2. $f\,g$
3. $f\,/\,g$ provided that $g \neq 0$, i.e. the function is continuous at all points x for which $g(x) \neq 0$.

Continuous Functions (5)

Lemma

Assume that f is continuous at $g(x_0)$, g continuous at x_0 and that $f \circ g$ is defined. Then

$$\lim_{x \to x_0} f \circ g(x) = \lim_{x \to x_0} f(g(x)) = f\left(\lim_{x \to x_0} g(x)\right) = f(g(x_0)).$$

Corollary

Assume that f is continuous at $g(x_0)$, g continuous at x_0 and that $f \circ g$ is defined. Then $f \circ g$ is continuous at $x = x_0$.

Continuous Functions (6)

Example

Function which is continuous at irrational points and discontinuous at rational points.

Represent rational numbers as $\frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}$ and m and n do not have common factors other than 1.

Define

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \\ 1 & \text{if } x = 0 \end{cases}$$

Continuous Functions (7)

Define

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \dfrac{1}{n} & \text{if } x = \dfrac{m}{n} \\ 1 & \text{if } x = 0 \end{cases}$$

The function f defined in this way is clearly discontinuous at rational points $\frac{m}{n}$ since $f\left(\frac{m}{n}\right) = \frac{1}{n} > 0$ and there are, arbitrarily close to the point $\frac{m}{n}$, irrational points x where $f(x) = 0$.

Continuity at irrational points follows from the fact that when approximating an irrational number by a rational number m/n, the denominator n grows arbitrarily large as the approximation gets better.

Bolzano's Theorem (1)

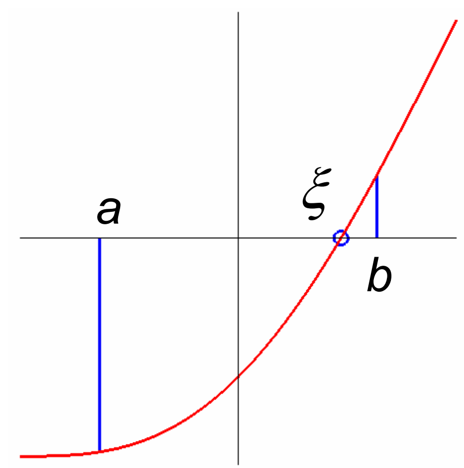
Bolzano's Theorem Assume that the function f is continuous on an interval $[a,b]$, $a < b$, and that $f(a) < 0$ and $f(b) > 0$. Then there is a point $\xi \in (a,b)$ such that $f(\xi) = 0$.

Proof Let $M = \{x \in [a,b] \mid f(x) < 0\}$.

Since, by the assumptions, $f(a) < 0$, $a \in M$, and hence $M \neq \emptyset$.

By the construction of the set M, $\forall x \in M : x < b$.

Hence M is non-empty and bounded from the above. This implies that $\xi = \sup(M)$ exists and that $\xi \in (a,b)$.



Bolzano's Theorem (2)

Bolzano'sTheorem

Assume that the function f is continuous on an interval $[a,b]$, $a<b$, and that $f(a)<0$ and $f(b)>0$. Then there is a point $\xi\in(a,b)$ such that $f(\xi)=0$.

Proof (cont'd)

Let $\xi=\sup(M)=\sup(\{x\in[a,b] \mid f(x)<0\})$. Claim: $f(\xi)=0$.

To prove the claim assume that it is not true, i.e. assume that $f(\xi)\neq 0$.

If $f(\xi)>0$, then $\exists\delta>0$ such that $|x-\xi|<\delta\Rightarrow f(x)>0$.

This means that $\xi=\sup(M)$ is not the smallest upper bound for the set M, which is a contradiction. Hence $f(\xi)\leq 0$.

If $f(\xi)<0$, then we see, exactly in the same way, that ξ is not an upper bound for the set M. This is also a contradiction. We conclude that $f(\xi)=0$. ■

Intermediate Value Theorem for Continuous Functions

Theorem Assume that the function f is continuous on an interval $[a,b]$, $a<b$, and that the number c is between $\mathrm{f}(a)$ and $\mathrm{f}(b)$. Then there is a number ξ between a and b such that $\mathrm{f}(\xi)=c$.

Proof If $c > \mathrm{f}(a)$, apply the previously shown Bolzano's Theorem to the function $\mathrm{f}(x) - c$.

Otherwise use the function $c - \mathrm{f}(x)$. ■

The Intermediate Value Theorem means that a function, continuous on an interval, takes any value between any two values that it takes on that interval. A continuous function cannot grow from being negative to positive without taking the value 0.

Using the Intermediate Value Theorem (1)

Problem Show that the equation $\cos(x) - 2x = 0$ has a solution.

Find an approximation of the solution with error < 0.001.

Solution Consider the function $f(x) = \cos(x) - 2x$.

With this notation: $\cos(x) - 2x = 0 \Leftrightarrow f(x) = 0$.

The function f is continuous. Since $f(0) = 1$ and $f(1) = \cos(1) - 2 < 0$, the Intermediate Value Theorem implies that there is $\xi \in (0,1)$ such that $f(\xi) = 0$. Hence the equation has a solution in the interval $(0,1)$.

Using the Intermediate Value Theorem (2)

Problem

Show that the equation $\cos(x) - 2x = 0$ has a solution.

Find an approximation of the solution with error < 0.001.

Solution (cont'd)

We now know that the solution ξ for the equation is in the interval $(0,1)$. To get a better approximation of the solution ξ, evaluate the function f at the midpoint of the interval $(0,1)$, i.e. at the point $\frac{1}{2}$.

Assume that $f\left(\frac{1}{2}\right) \neq 0$. Then if $f(0)f\left(\frac{1}{2}\right) < 0$, we know, by the Intermediate Value Theorem, that there is a solution in the interval $\left(0,\frac{1}{2}\right)$.

Otherwise $f(1)f\left(\frac{1}{2}\right) < 0$, and there is a solution in the interval $\left(\frac{1}{2},1\right)$.

Repeat the above to find an interval with length <0.002 containing the solution. The mid-point of this interval is the desired approximation.

Using the Intermediate Value Theorem (3)

Problem

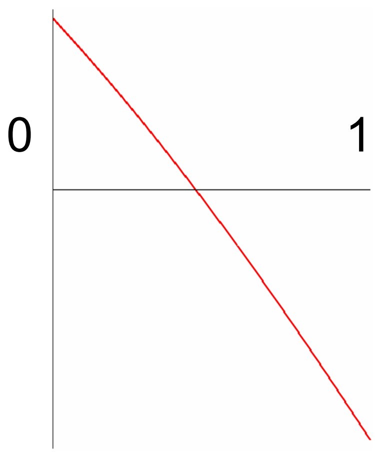
Show that the equation $\cos(x) - 2x = 0$ has a solution.

Find an approximation of the solution with error < 0.001.

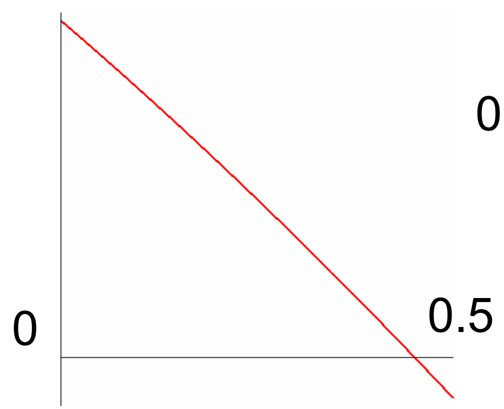
Graphical Solution

Look at the function $f(x) = \cos(x) - 2x$ first in the interval $(0,1)$.

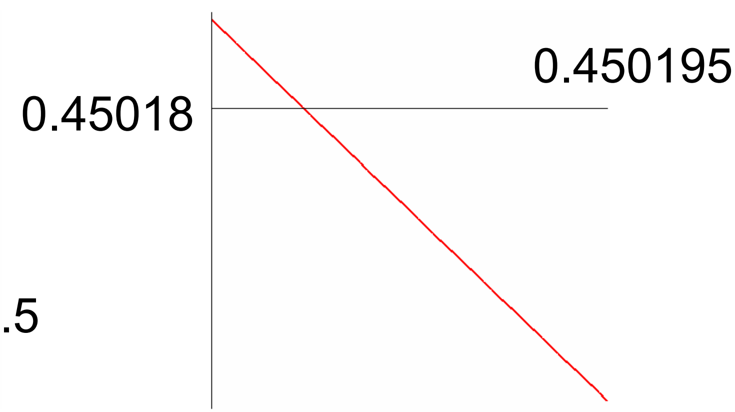
Make the interval smaller to approximate the solution. Repeat this to get the desired accuracy.



1st iteration, $\xi \approx 0.5$



2nd iteration, $\xi \approx 0.25$



17th iteration, $\xi \approx 0.45018750$