# Functions

Functions and their Graphs
Injectivity and Surjectivity
Limits of Functions
The Squeeze Theorem
Symmetric and Monotonous Functions

This Section deals with finite limits at finite points. Infinite limits and limits at the infinity are treated elsewhere.

# Definition of Functions

**Definition**

Given sets $A$ and $B$. A function $f : A \to B$ is a rule which assigns an element $f(a)$ of the set $B$ for every $a$ in $A$.

If the sets $A$ and $B$ are finite, then this rule can be expressed in terms of a table or a diagram.

Usually the sets $A$ and $B$ are not finite. In such a case the rule in question is usually expressed in terms of an algebraic expression, involving possibly special functions, for $f(a)$.

Alternatively the rule to compute $f(a)$, for a given $a$, may be a program taking $a$ as input and producing $f(a)$ as its output.

**Definition**

Let $f : A \to B$ be a function. The set $A$ is the domain of definition of the function $f$. The set $B$ is the target domain of the function $f$. The set $f(A) = \{ f(a) \mid a \in A \} \subset B$ is the range of the function $f$.

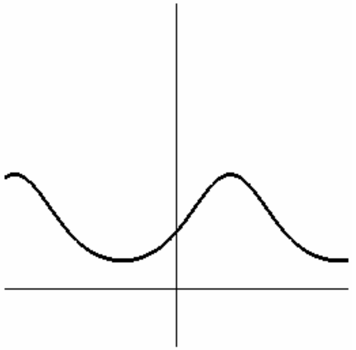
# Graphs of Functions

In calculus we are usually concerned with functions $f:\mathbb{R}\to\mathbb{R}$ defined in terms of explicit expressions for $f(x)$.

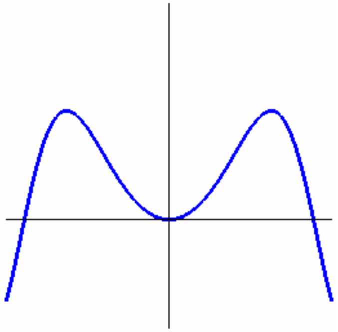
The product $\mathbb{R}^2=\{(x,y)\mid x,y\in\mathbb{R}\}$ is called the plane. It is usually pictured by drawing the $x$-axis horizontally and the $y$-axis vertically. The graph of a function $f:\mathbb{R}\to\mathbb{R}$ is the graph of the set $\{(x,f(x))\mid x\in\mathbb{R}\}$.

**Examples**

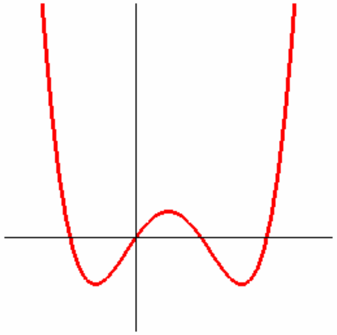
Below are the graphs of the functions $f(x)=\sin(x^2)$, $g(x)=x^4-2x^3-x^2+2x$, and $h(x)=2^{\sin(x)}$. Which is which?



$h(x)=2^{\sin(x)}$



$f(x)=\sin(x^2)$

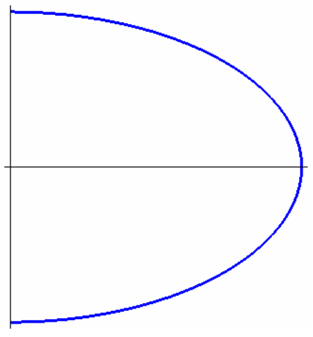


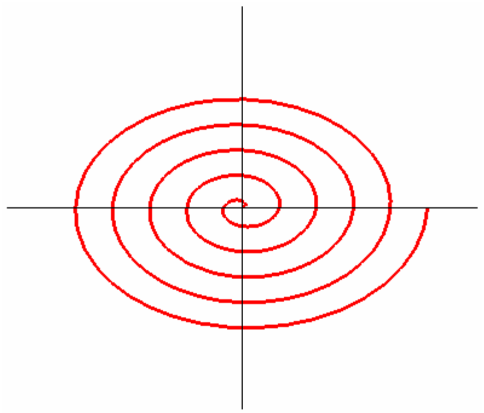
$g(x)=x^4-2x^3-x^2+2x$

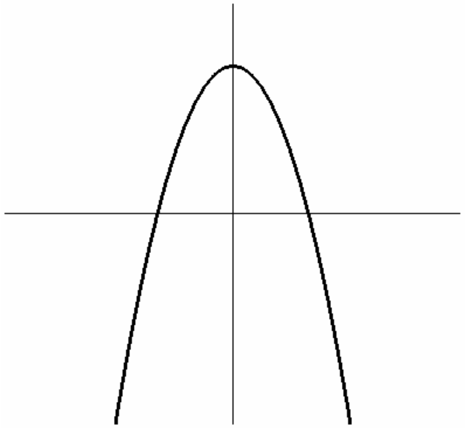
# Curves and Graphs

**Problem**

Which of the following curves in the plane are graphs of functions?







**Answer**

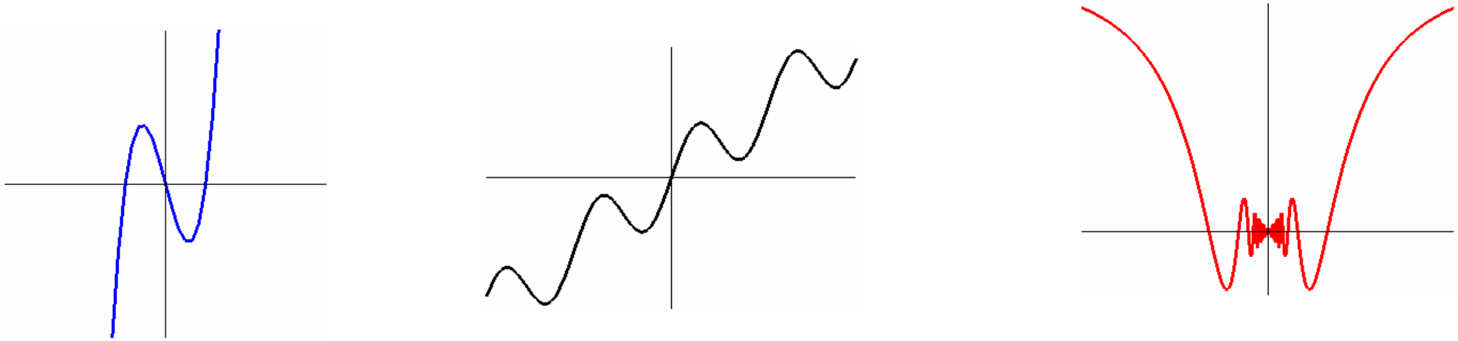
The first two curves are not graphs of functions since they do not correspond to a rule which associates a unique *y*-value to any given *x*-value. Graphically this means that there are vertical lines which intersect the first two curves at more than 1 point.

# Injective Functions

**Definition** A function $f : A \to B$ is injective or one-to-one if $f(x) = f(y) \Rightarrow x = y$.

A one-to-one function associates at most one point in the set $A$ to any given point in the set $B$.

**Problem** Which of the following graphs are graphs of one-to-one functions?



**Answer** None of the above graphs are graphs of one-to-one functions since they correspond to rules which associate several $x$-values to some $y$-values. This follows since there are horizontal lines intersecting the graphs at more than 1 point.

# Surjective Functions

**Definition**

A function $f : A \to B$ is surjective or onto if $f(A) = B$, i.e., if $\forall y \in B : \exists x \in A$ such that $f(x) = y$.

**Definition**

A function $f : A \to B$ is bijective if it is both one-to-one and surjective. For a bijective function f, $\forall y \in B : \exists! x \in A$ such that $f(x) = y$.

The notation "$\exists! x \in A$" means that "there is a unique element $x$ in the set $A$" having the specified property.

Observe that the property of being surjective or onto depends on how the set $B$ in the above is defined. Possibly reducing the set B any mapping f: $A \to B$ can always be made surjective.

# Limits of Functions

**Definition**

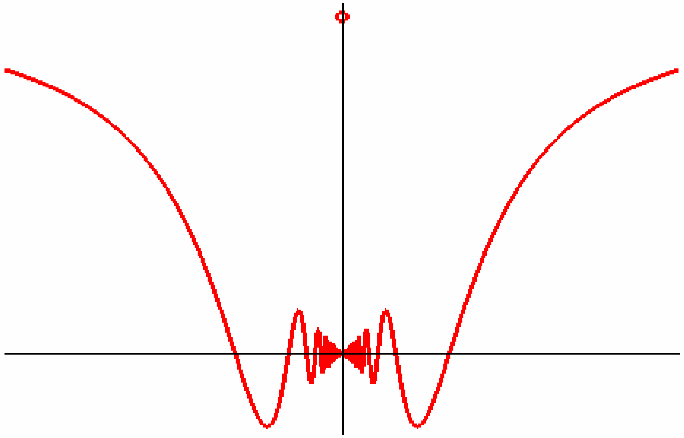
A function $f:\mathbb{R}\to\mathbb{R}$ has the limit $L$ at a point $x_0$ if the values $f(x)$ get arbitrarily close to $L$ as $x$ gets close to $x_0$ (but is not $x_0$).

**Example**

The function

$$f=\begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \\ 1 & \text{if } x=0 \end{cases}$$

has the limit $0$ at $x=0$ even if $f(0)=1$.



**Notation**

$$\lim_{x\to x_0} f(x)=L$$

# Formal Definition of Limits

**Definition** A function $f:\mathbb{R}\rightarrow\mathbb{R}$ has the limit $L$ at a point $x_0$

$$\forall\varepsilon>0:\exists\delta>0 \text{ such that } 0<\left|x-x_0\right|<\delta\Rightarrow\left|f(x)-L\right|<\varepsilon.$$

**Example**

**Claim** $$\lim_{x\to 0}\frac{1}{1+x^2}=1$$

**Proof** Let $\varepsilon>0$.

$$\left|1-\frac{1}{1+x^2}\right|=\left|\frac{x^2}{1+x^2}\right|\leq x^2<\varepsilon \text{ if } \left|x-0\right|<\sqrt{\varepsilon}=\delta. \qquad ■$$

The proof is now complete since for any positive number ε we were able to find a positive number δ satisfying the condition of the definition.

# Properties of Limits

Assume that $\lim\limits_{x\to x_0} f(x) = a$ and $\lim\limits_{x\to x_0} g(x) = b$, and let $c \in \mathbb{R}$.

1 $$\lim_{x\to x_0}\left(f(x)+g(x)\right) = a+b$$

2 $$\lim_{x\to x_0}\left(c\,f(x)\right) = ca$$

3 $$\lim_{x\to x_0}\left(f(x)g(x)\right) = ab$$

4 $$\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) = \frac{a}{b}$$ provided that $b \neq 0$.

**The Squeeze Theorem**

Assume further that near the number $x_0$, $f(x) \le h(x) \le g(x)$.

5 If $a = b$, i.e. if $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x)$,

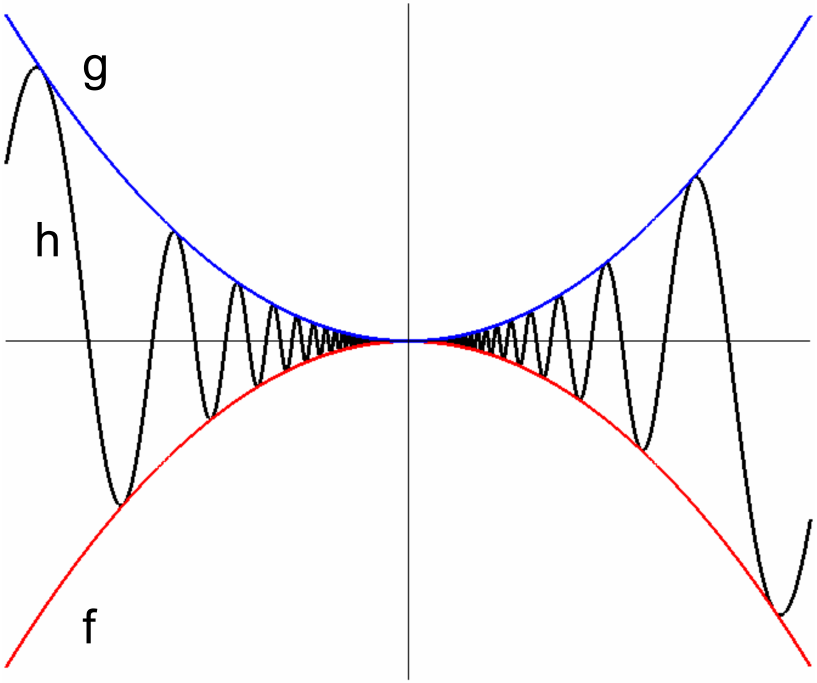
then $\lim\limits_{x\to x_0} h(x)$ exists and $\lim\limits_{x\to x_0} h(x) = \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x)$.

# Squeeze Theorem Graphically

**The Squeeze Theorem**

Assume that near the number $x_0$ the functions f, g and h satisfy $f(x) \leq h(x) \leq g(x)$.

If $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x)$, then $\lim\limits_{x \to x_0} h(x)$ exists and $\lim\limits_{x \to x_0} h(x) = \lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x)$.



In the Squeeze Theorem the values of the function h near the point $x_0$ are squeezed between the values of the functions f and g. If these functions have the same limit at $x_0$, then the function h must have that limit too. Observe that the values of f, g and h at $x_0$ need not be the limit value.

# How to Compute Limits (1)

Methods to compute limits:

1. If the function f is defined by an expression that has finite value at the limit point, then this finite value is the limit.
2. If the function f is defined by an expression whose value is undefined at the limit point, then one either has to rewrite the expression to a more suitable form or one has to use the Squeeze Theorem.

**Examples**

**1** $$\lim_{x\to 1}\frac{x-1}{1+x^2}=\frac{1-1}{1+1^2}=0$$

**2** $$\lim_{x\to 1}\frac{\sin\left(\frac{1}{x}\right)}{\sqrt{1+\cos^2(x)}}=\frac{\sin(1)}{\sqrt{1+\cos^2(1)}}$$

# How to Compute Limits (2)

**Example Requiring a Rewriting**

1

$$\lim_{x\to 0}\frac{x}{\sqrt{1+x}-\sqrt{1-x}}=\lim_{x\to 0}\frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\lim_{x\to 0}\frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}=\lim_{x\to 0}\frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{(1+x)-(1-x)}$$

$$=\lim_{x\to 0}\frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{2x}=\lim_{x\to 0}\frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)}{2}=1$$

# How to Compute Limits (3)

Application of the Squeeze Theorem

1 $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$ Observe that $-1 \le \sin(\alpha) \le 1$ for all $\alpha$.

Hence $-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$ for all $x$.

Since $\lim_{x\to 0}|x| = \lim_{x\to 0}(-|x|) = 0$ we can use the Squeeze Theorem.

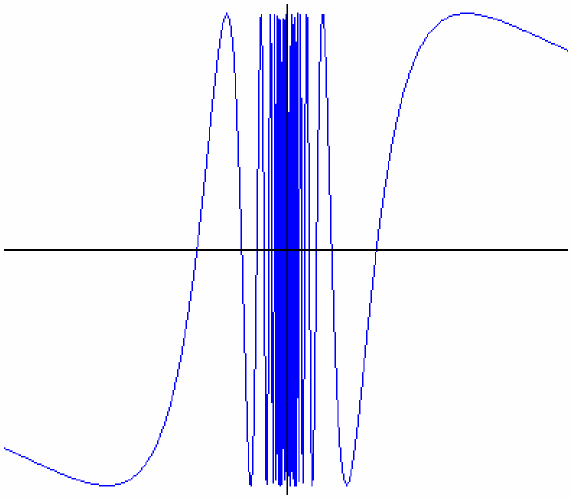
Squeeze Theorem $\Rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$

# No Limit

**Example** Let

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The function f does not have a limit at x=0 since arbitrarily close to x=0 the function f takes any value between -1 and 1.



# One-sided Limits (1)

**Definition** A function $f:\mathbb{R}\to\mathbb{R}$ has the left hand limit $L$ as $x$ approaches the point $x_0$ if the values $f(x)$ get arbitrarily close to $L$ as $x$ gets close to $x_0$ while $x<x_0$.

**Notation**
$$\lim_{x\to x_0-} f(x) = L$$

**Definition** A function $f:\mathbb{R}\to\mathbb{R}$ has the right hand limit $L$ as $x$ approaches the point $x_0$ if the values $f(x)$ get arbitrarily close to $L$ as $x$ gets close to $x_0$ while $x>x_0$.

**Notation**
$$\lim_{x\to x_0+} f(x) = L$$

# One-sided Limits (2)

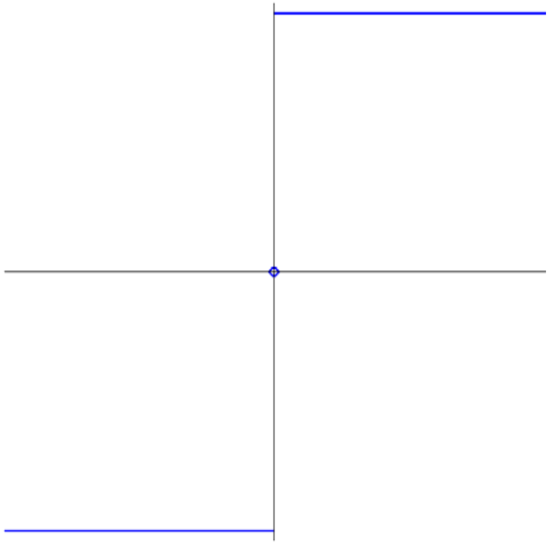
**Example**

Let

$$f(x)=\begin{cases}\dfrac{x}{|x|} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

$$\lim_{x\to 0-} f(x) = -1 \ \text{ and } \ \lim_{x\to 0+} f(x) = 1$$

The function f has one-sided limits at *x*=0 but does not have a limit at *x*=0.



# One-sided Limits (3)

**Definition**

A function $f:\mathbb{R}\to\mathbb{R}$ has the left hand limit $L$ as $x$ approaches the point $x_0$ if

$$\forall\varepsilon>0:\exists\delta>0 \text{ such that } 0<x_0-x<\delta\Rightarrow\left|f(x)-L\right|<\varepsilon.$$

**Definition**

A function $f:\mathbb{R}\to\mathbb{R}$ has the right hand limit $L$ as $x$ approaches the point $x_0$ if

$$\forall\varepsilon>0:\exists\delta>0 \text{ such that } 0<x-x_0<\delta\Rightarrow\left|f(x)-L\right|<\varepsilon.$$

**Lemma**

A function $f$ has a limit at $x=x_0$ if and only if both one sided limits exist and $\lim_{x\to x_0-}f(x)=\lim_{x\to x_0+}f(x)$.

The result follows immediately from the definitions. ■

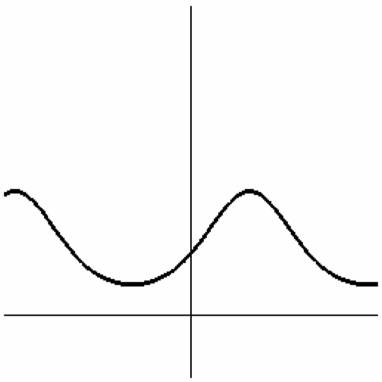
# Continuous Functions

**Definition**

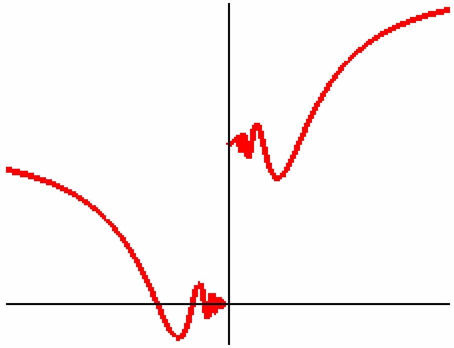
A function $f:\mathbb{R}\to\mathbb{R}$ is continous at $x=x_0$ if the limit $\lim_{x\to x_0} f(x)=f(x_0)$.

The function $f:\mathbb{R}\to\mathbb{R}$ is continous in an interval if it is continous at each point of the interval.

A function which is not continuous (at a point or in an interval) is said to be discontinuous.



Continuous function



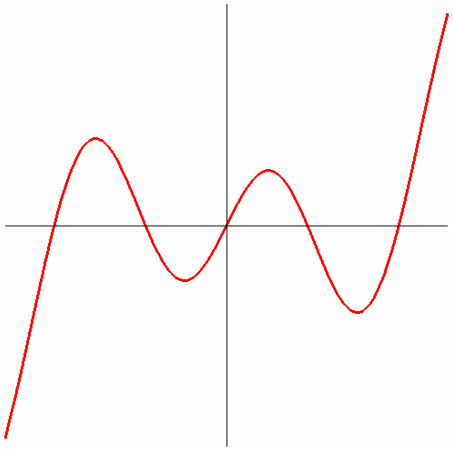
Discontinuous function

# Symmetric Functions (1)

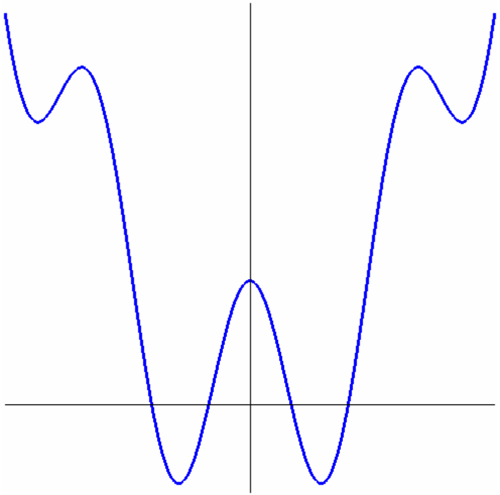
**Definition**

A function f is even if f(-$x$)=f($x$) for all $x$, and odd if f(-$x$)=-f($x$) for all $x$.

The above definition assumes that the domain of definition of the function f is symmetric, i.e. that if the function f is defined at a point x, then it is also defined at the point -x. In most of our applications, the functions under consideration are defined for all real numbers.



Odd function



Even function

# Symmetric Functions (2)

The property of being either odd or even can simplify greatly computations regarding a given function.

A polynomial is even (odd) if all of its terms have even (odd) power. Hence every polynomial is a sum of an even polynomial and an odd polynomial.

**Theorem**

Every function is a sum of an even function and an odd function assuming that the domain of definition of the function is symmetric.

**Proof**

Let f be a function. Define the functions $f_+$ and $f_-$ by setting

$$f_+(x) = \frac{f(x)+f(-x)}{2} \quad \text{and} \quad f_-(x) = \frac{f(x)-f(-x)}{2}.$$

$$f_+(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(-x)+f(x)}{2} = f_+(x), \text{ i.e. } f_+ \text{ is even.}$$

Similar computation shows that $f_-$ is odd.

Finally observe that $f = f_+ + f_-$. ∎

# Monotonous Functions

**Definition**

A function f is

1) increasing if $x_1 > x_2 \Rightarrow \mathrm{f}(x_1) > \mathrm{f}(x_2)$.

2) decreasing if $x_1 > x_2 \Rightarrow \mathrm{f}(x_1) < \mathrm{f}(x_2)$.

3) monotonous if it is either increasing or decreasing.

In some other texts, functions which are increasing in the above sense, are called strictly increasing. The same applies to the decreasing functions.

Like in the case of even and odd functions, any sufficiently smooth function can be expressed as a sum of an increasing function and a decreasing function. This is a deep fact of analysis.

Observe that monotonous functions are injective or one-to-one but that there are injective functions which are not monotonous.