Fundamental Theorem of Calculus

Basic Properties of Integrals

Through this section we assume that all functions are continuous on a closed interval $I = [a,b]$. Below r is a real number, f and g are functions.

Basic Properties of Integrals

1. $\int_c^c f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. $\int_a^b r\,f(x)dx = r\int_a^b f(x)dx$
5. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

These properties of integrals follow from the definition of integrals as limits of Riemann sums.

Upper and Lower Estimates

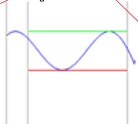
Theorem 1 If $f(x) \le g(x) \le h(x)\ \forall x \in [a,b]$,

$$\int_a^b f(x)dx \le \int_a^b g(x)dx \le \int_a^b h(x)dx.$$

Especially:

$$\min\{g(x) | x \in [a,b]\}(b-a) \le \int_a^b g(x)dx \le \max\{g(x) | x \in [a,b]\}(b-a).$$

The rectangle bounded from above by the red line is contained in the domain bounded by the graph of g.



The rectangle bounded from above by the green line contains the domain bounded by the graph of g.

Intermediate Value Theorem for Integrals

Theorem 2

$\exists \xi \in [a,b]$ such that $\int_a^b f(x)dx = f(\xi)(b-a)$.

Proof By the previous theorem,

$\min\{f(x) | x \in [a,b]\} \le \frac{1}{b-a} \int_a^b f(x)dx \le \max\{f(x) | x \in [a,b]\}$

By the Intermediate Value Theorem for Continuous Functions,

$\exists \xi \in [a,b]$ such that $f(\xi) = \frac{1}{b-a} \int_a^b f(x)dx$.

This proves the theorem. ■

First Part of the Fundamental Theorem of Calculus

First Part of the Fundamental Theorem of Calculus

The function $F(x) = \int_a^x f(t)dt$ is differentiable for $x \in (a,b)$ and $F'(x) = f(x)$ for all $x \in (a,b)$.

Proof Let $h \neq 0$. $\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$ (By the properties of integrals.)

$= \frac{1}{h}\left(f(\xi_h)\right)\left((x+h)-x\right) = f(\xi_h)$ where

ξ_h is between x and $x+h$. (By the Intermediate Value Theorem for Integrals)

As $h \to 0$, $\xi_h \to x$. Since f is continuous, $\lim_{h\to 0} f(\xi_h) = f(x)$. ■

Second Part of the Fundamental Theorem of Calculus

Second Part of the Fundamental Theorem of Calculus

Assume that F is an antiderivative of a continuous function f. Then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof By the First Fundamental Theorem of Calculus, the function

$G(x) = \int_a^x f(t)dt$ is an antiderivative of the function f.

We have $G(a) = 0$ and $\int_a^b f(x)dx = G(b) = G(b) - G(a)$.

If F is a general antiderivative of the function f, then $F(x) = G(x) + C$

for some constant C. Hence $F(b) - F(a) = G(b) - G(a) = \int_a^b f(x)dx$. ■

Fundamental Theorem of Calculus

We collect the previous two results into one theorem.

Fundamental Theorem of Calculus

Assume that f is a continuous function.

1. The function $g(x)=\int_a^x f(t)dt$ is an antiderivative of f.
2. Let F be an antiderivative of f. Then $\int_a^b f(x)dx = F(b)-F(a)$.

Notation $F(x)]_a^b = F(b)-F(a)$. Other common notations for the same quantity $F(x)|_a^b$ and $[F(x)]_a^b$.

We have $\int_a^b f(x)dx = F(x)]_a^b$.

Examples (1)

Example Let $f(x)=\int_0^x e^{-t^2}dt$. Compute $f'(x)$.

Solution

The function to be integrated in the formula defining f is continuous. Hence $f'(x)=e^{-x^2}$ by the Fundamental Theorem of Calculus.

Examples (2)

Example Let $g(x)=\int_0^{x^2}\frac{\sin(t)}{t}dt$. Compute $g'(x)$.

Solution

Here one must first observe that the function $h(t)=\frac{\sin(t)}{t}$ is everywhere continuous provided that we set $h(0)=1$. Hence the integral is well defined and we can apply the Fundamental Theorem of Calculus.

Let $f(u)=\int_0^u \frac{\sin(t)}{t}dt$, and $u(x)=x^2$. Then $f'(u)=\frac{\sin(u)}{u}$,

$$g(x)=f(u(x)) \Rightarrow g'(x)=f'(u(x))u'(x)=\frac{\sin(x^2)}{x^2}(2x)=\frac{2\sin(x^2)}{x}$$