Power Series

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Power Series

Definition A power series is a series of the type

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

Here x is a variable. Substituting a numeric value $x=x_0$ for the variable x one gets a regular series $P(x_0)$. If this regular series converges, then we say that the power series $P(x)$ converges at the point $x=x_0$.

Example The power series $P(x) = 0 + x + 2x^2 + 3x^3 + \ldots = \sum_{k=0}^{\infty} kx^k$

converges, by the Ratio Test, for all the values of x for which

$$\lim_{k\to\infty} \frac{\left|(k+1)x^{k+1}\right|}{\left|kx^k\right|} = \lim_{k\to\infty} \frac{k+1}{k}|x| = |x| < 1.$$

We conclude that the power series $P(x)$ converges for all values of x, $-1<x<1$. The series diverges otherwise.

Convergence of a Power Series

Theorem Assume that the power series $P(x) = \sum_{k=1}^{\infty} a_k x^k$ converges at some point $x = x_0 \neq 0$. The series $P(x)$ converges at any point $x = s$ satisfying $|s| < |x_0|$.

Proof Since the series $P(x_0) = \sum_{k=1}^{\infty} a_k x_0^k$ converges,

Since $\left|a_k x_0^k\right| < 1$

$\lim_{k\to\infty} a_k x_0^k = 0$. Hence $\exists k_1 : k > k_1 \Rightarrow \left|a_k x_0^k\right| < 1$. If $k > k_1$, $\left|a_k s^k\right| = \left|a_k x_0^k\right| \left|\frac{s}{x_0}\right|^k < \left|\frac{s}{x_0}\right|^k$.

Since $|s| \leq |x_0|$, $\left|\frac{s}{x_0}\right| < 1$. Therefore $\sum_{k=k_1}^{\infty} \left|\frac{s}{x_0}\right|^k$ is a converging geometric series.

We conclude that the series $\sum_{k=1}^{\infty} a_k s^k$ converges absolutely and hence converges. ■

Convergence of a Power Series

Remark If the power series $P(x)=\sum_{k=1}^{\infty}a_kx^k$ converges at some point $x=x_0\neq 0$ it does not need to converge at $x=-x_0$. For example, the Power Series $\sum_{k=1}^{\infty}\frac{x^k}{k}$ converges at $x=-1$ and diverges at $x=1$.

Observation The Power Series $P(x)=\sum_{k=0}^{\infty}a_kx^k$ defines a function whose domain of definition consists of all the points x at which $P(x)$ converges.

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Power Series as Functions

The Fibonacci Numbers F_n, $n = 0,1,2,\ldots$ are defined recursively by setting $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

The sequence of Fibonacci numbers starts $(0,1,1,2,3,5,8,\ldots)$.

Example Consider $P(x)=\sum_{k=0}^{\infty}F_kx^k$.

In an exercise we have seen that the series $P(x)$ converges for $x=1/2$.

One concludes that $P(x)=\sum_{k=0}^{\infty}F_kx^k$ is a function whose domain of definition contains the interval $\left(-\frac{1}{2},\frac{1}{2}\right]$.

Historical Link: Leonardo Pisano aka Leonardo Fibonacci (c1175 – 1250)

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Generating Function for Fibonacci Numbers

Example Consider $P(x)=\sum_{k=0}^{\infty}F_kx^k$.

The following computation is valid at least in $\left(-\frac{1}{2},\frac{1}{2}\right]$:

$$P(x)=F_0+F_1x+F_2x^2+F_3x^3+F_4x^4+\cdots$$

$$xP(x)=F_0x+F_1x^2+F_2x^3+F_3x^4+\cdots$$

$$x^2P(x)=F_0x^2+F_1x^3+F_2x^4+\cdots$$

All these terms are 0 by the definition of the Fibonacci numbers.

$$P(x)-xP(x)-x^2P(x)=$$

$$F_0+(F_1-F_0)x+(F_2-F_1-F_0)x^2+(F_3-F_2-F_1)x^3+(F_4-F_3-F_2)x^4+\cdots$$

Hence $P(x)-xP(x)-x^2P(x)=x$.

Conclude that $P(x)=\sum_{k=0}^{\infty}F_kx^k=\frac{x}{1-x-x^2}$.

Definition The function $P(x)$ is the generating function for Fibonacci numbers.

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Radius of Convergence

Consider a general power series $P(x)=\sum_{k=1}^{\infty} a_k x^k$.

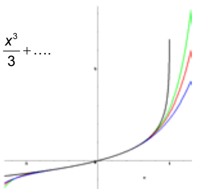
By the Ratio Test this series converges if $\lim_{k\to\infty}\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right|<1$.

Compute in the following way:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right|=\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right||x|=|x|\left[\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|\right]<1\Leftrightarrow|x|<\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|.$$

Definition The limit $R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$ is the radius of convergence of the power series $P(x)$ provided that the limit exists. The power series $P(x)$ converges for $|x|<R$ and diverges for $|x|>R$. At the points $x=R$ and $x=-R$ the series may converge or diverge.

Example



Consider the power series $P(x)=\sum_{k=1}^{\infty}\frac{1}{k}x^k=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots.$

The radius of convergence of this series is

$$R=\lim_{k\to\infty}\frac{\left|\frac{1}{k}\right|}{\left|\frac{1}{k+1}\right|}=\lim_{k\to\infty}\left|\frac{k+1}{k}\right|=1.$$

One concludes that the power series $P(x)$ converges if $|x|<1$, and diverges if $|x|>1$. The series $P(1)$ is the harmonic series, i.e., it diverges. The series $P(-1)$ is a convergent alternating series.

We will later see that, in the interval $(-1,1)$, the power series represents the function $f(x)=-\ln(1-x)$. The above plot shows the graph of the function f together with the graphs of the partial series

$P_m(x)=\sum_{k=1}^{m}\frac{x^k}{k}$, for $m=5$ (blue), $m=7$ (red) and $m=9$ (green).

Finding Power Series

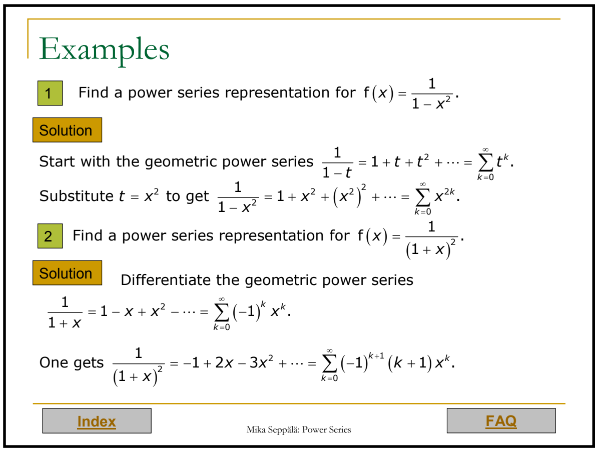
Geometric Power Series $\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{k=0}^{\infty}x^k$

The Geometric Power Series converges for $|x|<1$ and diverges otherwise.

Using power series representations for known functions one can derive power series respresentation for other functions by the following tricks:

1. substitution,
2. differentiation,
3. integration.

Within the interval of convergence, powers series can be differentiated and integrated term by term.



Examples

1 Find a power series representation for $f(x) = \frac{1}{1-x^2}$.

Solution

Start with the geometric power series $\frac{1}{1-t} = 1 + t + t^2 + \cdots = \sum_{k=0}^{\infty} t^k$.

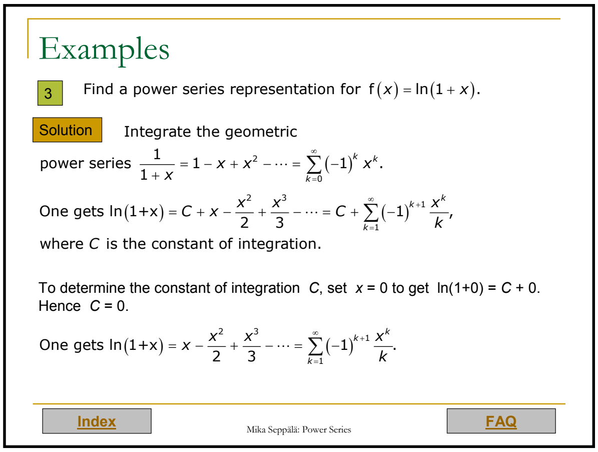
Substitute $t = x^2$ to get $\frac{1}{1-x^2} = 1 + x^2 + \left(x^2\right)^2 + \cdots = \sum_{k=0}^{\infty} x^{2k}$.

2 Find a power series representation for $f(x) = \frac{1}{(1+x)^2}$.

Solution Differentiate the geometric power series

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots = \sum_{k=0}^{\infty} (-1)^k x^k.$$

One gets $\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + \cdots = \sum_{k=0}^{\infty} (-1)^{k+1} (k+1) x^k$.



Examples

3 Find a power series representation for $f(x) = \ln(1+x)$.

Solution Integrate the geometric

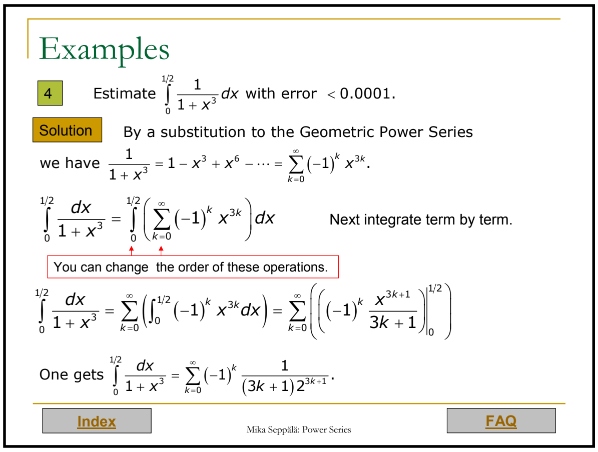
power series $\frac{1}{1+x} = 1 - x + x^2 - \cdots = \sum_{k=0}^{\infty} (-1)^k x^k$.

One gets $\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = C + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$,

where C is the constant of integration.

To determine the constant of integration C, set $x = 0$ to get $\ln(1+0) = C + 0$. Hence $C = 0$.

One gets $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$.



Examples

4 Estimate $\int_0^{1/2} \frac{1}{1+x^3} dx$ with error < 0.0001.

Solution By a substitution to the Geometric Power Series

we have $\frac{1}{1+x^3} = 1 - x^3 + x^6 - \cdots = \sum_{k=0}^{\infty} (-1)^k x^{3k}$.

$$\int_0^{1/2} \frac{dx}{1+x^3} = \int_0^{1/2} \left(\sum_{k=0}^{\infty} (-1)^k x^{3k} \right) dx$$

Next integrate term by term.

You can change the order of these operations.

$$\int_0^{1/2} \frac{dx}{1+x^3} = \sum_{k=0}^{\infty} \left(\int_0^{1/2} (-1)^k x^{3k} dx \right) = \sum_{k=0}^{\infty} \left(\left((-1)^k \frac{x^{3k+1}}{3k+1} \right) \Bigg|_0^{1/2} \right)$$

One gets $\int_0^{1/2} \frac{dx}{1+x^3} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(3k+1) 2^{3k+1}}$.

Examples

4 Estimate $\int_0^{1/2} \frac{1}{1+x^3}\,dx$ with error < 0.0001.

Solution (cont'd) We have obtained $\int_0^{1/2} \frac{dx}{1+x^3} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(3k+1)2^{3k+1}}$.

When approximating the sum of this alternating series by finite partial sums, the error is less than the absolute value of the first term left out.

Direct computation yields $(3k + 1)2^{3k+1} = 10240$ if $k = 3$.

The value $k = 3$ corresponds to the fourth term of the above alternating series since summations starts from $k = 0$. Hence it suffices to compute the first three terms of the above alternating series for the integral in question.

$$\int_0^{1/2} \frac{dx}{1+x^3} \approx \sum_{k=0}^{2} (-1)^k \frac{1}{(3k+1)2^{2k+1}} \approx 0.48549$$

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