The Integral Test and The Harmonic Series

Let $a_k = \mathrm{f}(k)$, where f is a decreasing function taking only positive values. If the improper integral $\int_1^{\infty} \mathrm{f}(x)\,dx$ converges then also the series $\sum_{k=1}^{\infty} a_k$ converges.

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Integral Test

Theorem Let $a_k = f(k)$, where f is a decreasing function taking only positive values. If the improper integral $\int_1^\infty f(x)dx$ converges then also the series $\sum_{k=1}^{\infty} a_k$ converges, and $\int_1^\infty f(x)dx \le \sum_{k=1}^{\infty} a_k \le \int_0^\infty f(x)dx.$

Proof Observe that $S_M = \sum_{k=1}^{M} a_k$ is a right rule approximation for the integral $\int_0^M f(x)dx.$

Since f is positive and decreasing $S_M = \sum_{k=1}^{M} a_k \le \int_0^M f(x)dx \le \int_0^\infty f(x)dx.$

We conclude that (S_M) is a bounded increasing sequence. Hence it has a finite limit.

This means that the series converges. ∎

Integral Test

Integral Test Let $a_k = \mathrm{f}(k)$, where f is a decreasing function taking only positive values. If the improper integral $\int_1^\infty \mathrm{f}(x)dx$ converges then also the series $\sum_{k=1}^\infty a_k$ converges, and $\int_1^\infty \mathrm{f}(x)dx \le \sum_{k=1}^\infty a_k \le \int_0^\infty \mathrm{f}(x)dx$. If the improper integral $\int_1^\infty \mathrm{f}(x)dx$ diverges, then also the series $\sum_{k=1}^\infty a_k$ diverges.

Proof By the previous Theorem it suffices to show the divergence part only.

If $\int_1^\infty \mathrm{f}(x)dx$ diverges, then, since f is a non-negative function,

$\lim_{M\to\infty} \int_1^M \mathrm{f}(x)dx = \infty$. Since $\int_1^M \mathrm{f}(x)dx \le \sum_{k=1}^M a_k$ we also have $\lim_{M\to\infty} \sum_{k=1}^M a_k = \infty$. ■

Error Estimates by the Integral Test

Let $a_k = \mathrm{f}(k)$, where f is a decreasing function taking only positive values. The estimate $\int_1^\infty \mathrm{f}(x)dx \le \sum_{k=1}^{\infty} a_k \le \int_0^\infty \mathrm{f}(x)dx$ of the Integral Test can be generalized as $\int_{M+1}^\infty \mathrm{f}(x)dx \le \sum_{k=M+1}^{\infty} a_k \le \int_M^\infty \mathrm{f}(x)dx.$

This follows directly from the argument of the proof of the Integral Test.

Hence the error made when approximating the sum $\sum_{k=1}^{\infty} a_k$ by $\sum_{k=1}^{M} a_k$ satisfies $\left|\sum_{k=1}^{M} a_k - \sum_{k=1}^{\infty} a_k\right| = \left|\sum_{k=M+1}^{\infty} a_k\right| \le \int_M^\infty \mathrm{f}(x)dx.$

Here we assume that the series converges.

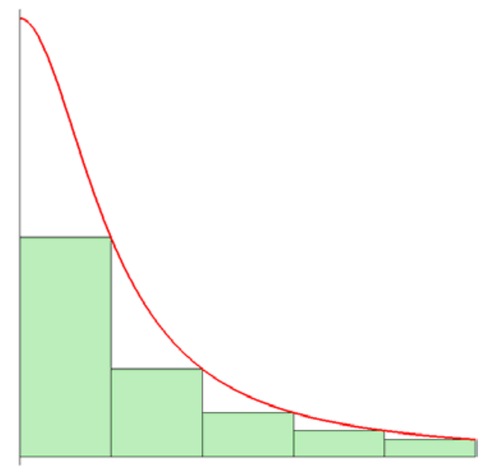
Error Estimate

$$\left|\sum_{k=1}^{M} a_k - \sum_{k=1}^{\infty} a_k\right| \le \int_M^\infty \mathrm{f}(x)dx$$

Example of the Usage of the Integral Test

Example The series $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$ converges since $\int_0^{\infty} \frac{1}{1+x^2} dx$ converges.

In the picture on the right the areas of the green boxes correspond to the elements of the series $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$. The red curve is the graph of the function $\frac{1}{1+x^2}$. We know that



$$\int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{M\to\infty} \int_0^{M} \frac{1}{1+x^2} dx = \lim_{M\to\infty} \arctan(x)\Big|_0^M = \frac{\pi}{2}.$$

The area under the red curve is finite. This implies that the sum of the areas of the green boxes is also finite, i.e., that the series in question converges.

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Harmonic Series

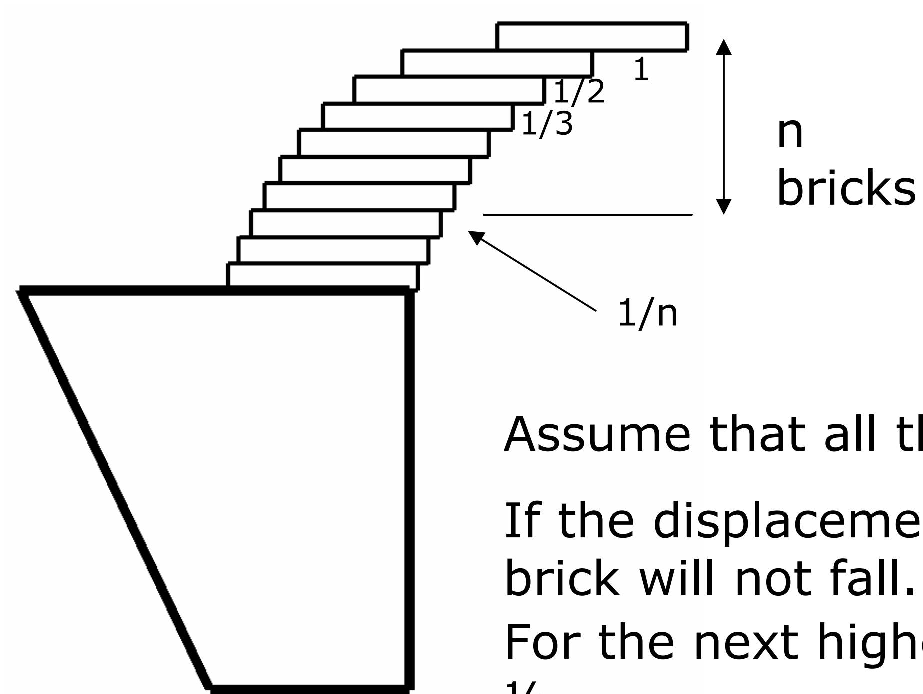
Definition The series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is called the harmonic series.

Let $S_M = \sum_{k=1}^{M} \frac{1}{k}.$ The following table of values has been computed by Maple

S_4	=	2.0833...	S_{10}	=	2.9289...
S_{100}	=	5.1873...	S_{1000}	=	7.4854...
S_{10000}	=	9.7876...	S_{100000}	=	12.0901...
$S_{1000000}$	=	14.3927...	$S_{10000000}$	=	16.6953...

These numeric computation appear to suggest that the harmonic series converges. The Integral Test proves, however, that the harmonic series diverges since the corresponding improper integral diverges.

Application of Harmonic Series



To build a cupola the builder lays bricks above each other and displaces them as far to the right as possible without making the building collapse. This is indicated in the picture on the left.

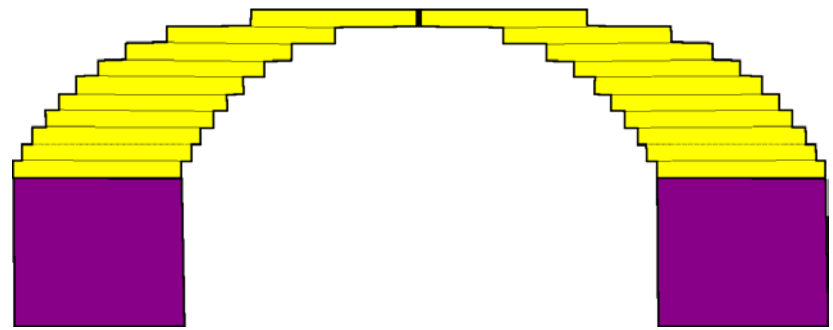
Assume that all the bricks have the length 2 units.

If the displacement of the highest brick is 1 unit, the brick will not fall.

For the next highest brick the displacement can be ½.

One can continue this inductively. The displacement of the n^{th} highest brick can be 1/n. Since the harmonic series diverges, one can build arbitrarily wide cupola in this way.

Cathedral of Florence and the Cupola of Brunelleschi



Since the harmonic series diverges, one could build arbitrarily large cupola whose stability is based on this fact. Prime example of a large cupola is the cupola of the cathedral of Florence. This cupola was designed by Brunelleschi. The cupola was completed in 1436. The shape and the width of the cupola of Brunelleschi suggest, however, that its stability is not based on the divergence of the harmonic series.