

The Ratio Test and The Root Test

- 1** Suppose that $\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = q$. **2** Suppose that $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = r$.
- The series $\sum_{k=1}^{\infty} a_k$ converges if $q < 1$ and diverges if $q > 1$. The series $\sum_{k=1}^{\infty} a_k$ converges if $r < 1$ and diverges if $r > 1$.

The Ratio Test and The Root Test

Ratio Test Suppose that $\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = q$. The series $\sum_{k=1}^{\infty} a_k$ converges if $q < 1$ and diverges if $q > 1$. The test is inconclusive if $q=1$.

Proof Assume that $\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = q < 1$. Then for any number ρ , $q < \rho < 1$, there is a number k_p such that $k > k_p \Rightarrow \frac{|a_{k+1}|}{|a_k|} < \rho$. It suffices to show that $\sum_{k=k_p+1}^{\infty} a_k$ converges. Observe now that $|a_k| < \rho |a_{k-1}| < \dots < \rho^{k-k_p} |a_{k_p}|$. Since $|\rho| = \rho < 1$, the geometric series $\sum_{k=k_p+1}^{\infty} \rho^{k-k_p-1} |a_{k_p}|$ converges.

Mika Seppälä: Ratio Test

Ratio Test

Ratio Test Suppose that $\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = q$. The series $\sum_{k=1}^{\infty} a_k$ converges if $q < 1$ and diverges if $q > 1$. The test is inconclusive if $q=1$.

Proof (cont'd) Since $|a_k| \leq \rho^{k-k_p} |a_{k_p}|$ and since the geometric series $\sum_{k=k_p+1}^{\infty} \rho^{k-k_p-1} |a_{k_p}|$ converges, the series $\sum_{k=k_p+1}^{\infty} |a_k|$ converges by the Comparison Test. This means that $\sum_{k=k_p+1}^{\infty} a_k$ converges absolutely. Hence the series $\sum_{k=1}^{\infty} a_k$ converges. If $q > 1$, the series $(|a_k|)$ is increasing after some k_1 . Hence $\lim_{k \rightarrow \infty} a_k \neq 0$ and the series $\sum_{k=1}^{\infty} a_k$ diverges. ■

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Usage of the Ratio Test

Example 1 Show that the series

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k} \text{ converges.}$$

Solution Use the Ratio Test.

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{(k+1)^2}{2^{k+1}}}{\frac{k^2}{2^k}} = \frac{(k+1)^2}{k^2} \cdot \frac{2^k}{2^{k+1}} = \left(\frac{k+1}{k}\right)^2 \frac{1}{2} \xrightarrow{k \rightarrow \infty} \frac{1}{2}$$

Hence the series converges by the Ratio Test.

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Usage of the Ratio Test

Example 2 Show that the series $\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$ converges.

Solution Use the Ratio Test.

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)!} = \frac{1}{k+1} \xrightarrow{k \rightarrow \infty} 0$$

Hence the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ converges. ■

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The Ratio Test

The same arguments that were used to prove the convergence part of the Ratio Test can be used to show also the following generalization:

Ratio Test Suppose that $\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = q$. The series $\sum_{k=1}^{\infty} a_k$ converges

if $|q| = q < 1$ and diverges if $q > 1$.

Remark 1 The alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges by the Alternating

Series Test but we have

$$\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \rightarrow \infty} \frac{\frac{(-1)^{k+2}}{k+1}}{\frac{(-1)^{k+1}}{k}} = \lim_{k \rightarrow \infty} \frac{k}{k+1} = 1.$$

Mika Seppälä: Ratio Test

The Ratio Test

Ratio Test Suppose that $\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = q$. The series $\sum_{k=1}^{\infty} a_k$ converges if $|q| < 1$ and diverges if $|q| > 1$.

Remark 2 The Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the Integral

Test but we have

$$\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \rightarrow \infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{k}{k+1} = 1.$$

Conclusion If the limit q in the Ratio Test equals 1, we cannot conclude anything about the convergence of the series in question. It may converge or diverge.

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The Root Test

Root Test Suppose that $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = r$. The series $\sum_{k=1}^{\infty} a_k$ converges if $r < 1$ and diverges if $r > 1$. If $r = 1$, the test is inconclusive.

Remark The Harmonic Series and its associated Alternating Series serve also here as examples of the fact that the Root Test is inconclusive if $r = 1$.

The proof of the Root Test is similar to that of the Ratio Test. If $r < 1$, we can again here compare the series in question to a convergent geometric series. The (absolute) convergence of the series then follows from that of the geometric series. Details are left as an exercise.

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Usage of the Root Test

Example Does the series $\sum_{k=1}^{\infty} \left(\frac{2k+3}{4k+5}\right)^k$ converge or diverge?

Solution Use the Root Test.

For the series $\sum_{k=1}^{\infty} \left(\frac{2k+3}{4k+5}\right)^k$ we have $a_k = \left(\frac{2k+3}{4k+5}\right)^k$.

$$\text{Hence } \sqrt[k]{|a_k|} = \sqrt[k]{\left(\frac{2k+3}{4k+5}\right)^k} = \frac{2k+3}{4k+5} = \frac{2+3/k}{4+5/k} \xrightarrow{k \rightarrow \infty} \frac{2}{4} = \frac{1}{2} < 1.$$

We conclude that the series in the example converges by the Root Test.

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