Mathematical thinking

Mathematics is not difficult – or is it?

Justifications of mathematical facts are always series of simple deductions one after another. One has to take, as a starting point, a known mathematical fact and try to find a sequence of simple steps which show that the desired result follows from this known fact. In a sense mathematics is easy. It is not necessary to memorize a large number of facts by heart. One simply has to use all known information as well as possible to derive new information or to solve problems. This using of all known information is often difficult. It is easy to overlook some of the given information.

Proofs by induction

One can use mathematical induction to show that a statement $S(n)$ that depends on a natural number n is true for all possible values of n.

Mathematical induction has the following steps:

1. Show first that the statement $S(n)$ is true for the first value (usually 1) of the parameter n.
2. Next make the Induction Assumption: the statement is true for $n = m$.
3. Finally show that the above implies that the statement is also true for $n = m + 1$.

Remark (Induction as a statement proving machine).
Mathematical Induction can be thought of like a machine that eventually proves the statement in question for all (positive integer) values of the parameter.

Examples

1. Assuming that $q \neq 0,1$, show that $\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$

   The proof by induction of this fact has the following steps:

   a. The first value of n, for which we have to verify the above formula, is 0. We get
   $$\sum_{k=0}^{0} q^k = q^0 = 1 = \frac{1-q^{0+1}}{1-q}.$$
   b. Next we make the induction assumption, and assume that
   $$\sum_{k=0}^{m} q^k = \frac{1-q^{m+1}}{1-q}.$$
   c. The proof is complete when we show that, from the above induction assumption it follows that

$$\sum_{k=0}^{m+1} q^k = \frac{1-q^{m+2}}{1-q}.$$

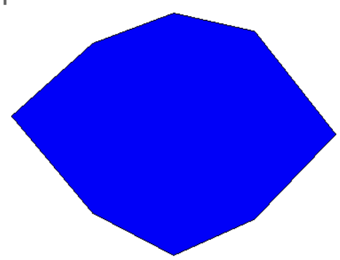
This is immediate since, by the induction assumption,

$$\sum_{k=0}^{m+1} q^k = \left(\sum_{k=0}^{m} q^k\right) + q^{m+1} = \frac{1-q^{m+1}}{1-q} + q^{m+1} = \frac{1-q^{m+2}}{1-q}.$$

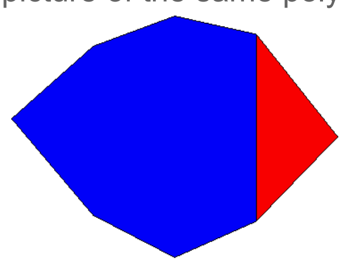
2. Show that the sum of angles of a convex polygon with $n+2$ sides is $n\pi$ (radians).

   The proof by induction of this fact has the following steps:

   a. The first value of n for which we have to show the correctness of the formula is 1. A convex polygon with 3 sides is a triangle. It is known that the sum of the angles of a triangle is π radians.
   b. Next assume that the sum of the angles of a convex polygon with $m+2$ sides is $m\pi$. This is the Induction Assumption.
   c. Consider a polygon with $(m+1)+2 = m+3$ sides as in the first picture below.

      From that polygon, cut away a triangle as indicated in the following picture of the same polygon.

      In this way the original polygon with $m+3$ sides is decomposed into a triangle and into a convex polygon with $m+2$ sides. The sum of the angles of the polygon with $m+3$ sides equals the sum of the angles of the red (gray if you see this page in b/w colors) triangle (π) plus the sum of the angles of the blue (black in b/w) polygon with $m+2$ sides (in the second picture above, by the induction assumption this sum is $m\pi$). Hence the sum of the angles of a convex polygon with $m+3$ sides is $\pi + m\pi = (m+1)\pi$. This completes the proof of this statement.

Exercises

Show the following formulae by induction.

1. $\sum_{n=1}^{m} n = \frac{m^2}{2} + \frac{m}{2}$
2. $\sum_{n=1}^{m} n^2 = \frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6}$