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# Solved Problems about Power Series

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# Summary of Power Series

Power Series

$$S(x) = \sum_{k=0}^{\infty} a_k x^k, \text{ assume } a_k \neq 0 \text{ for all } k.$$

Basic Power Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{k=0}^{\infty} x^k$$

By the Ratio Test

Radius of Convergence

$$R = \lim_{k \rightarrow \infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k \rightarrow \infty} \frac{1}{\sqrt[k]{|a_k|}}$$

By the Root Test

The Power Series  $S(x)$  converges if  $|x| < R$  and diverges if  $|x| > R$ .

Differentiation

$$S'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

The swapping of summation and integration is valid in the interval of convergence.

Integration

$$\int S(x) dx = \int \left( \sum_{k=0}^{\infty} a_k x^k \right) dx = \sum_{k=0}^{\infty} \left( \int a_k x^k dx \right) = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$$
$$\int_a^b S(x) dx = \int_a^b \left( \sum_{k=0}^{\infty} a_k x^k \right) dx = \sum_{k=0}^{\infty} \left( \int_a^b a_k x^k dx \right) = \sum_{k=0}^{\infty} \left( \frac{a_k}{k+1} x^{k+1} \Big|_a^b \right)$$

# Overview of Problems

1 Show that the series  $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$  converges for  $|x| < \frac{1}{2}$ .

2 Find the radius of convergence of  $\sum_{n=1}^{\infty} \frac{(2n)!x^n}{(n!)^2}$ .

In Problems 3 and 4, the radius of convergence of the power series  $\sum_{k=0}^{\infty} a_k x^k$  is  $R$ .

3 What is the radius of convergence of the power series  $\sum_{k=1}^{\infty} k a_k x^{k-1}$ ?

4 What is the radius of convergence of the power series  $\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ ?

5 Find a power series representation for the function  $f(x) = \frac{x^2}{1-2x}$

and determine the radius of convergence of the power series.

# Overview of Problems

6 Find the radius of convergence of the power series

$$\frac{x}{1-x-x^2} = F_0 + F_1x + F_2x^2 + \dots \text{ where } F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

are the Fibonacci numbers,  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$ .

7 Find power series representations for the

$$\text{functions } f(x) = \frac{1}{1-x^2} \text{ and } g(x) = \frac{x^2}{x+1}.$$

8 Find power series representations for the integrals  $\int \frac{dx}{1-x}$  and  $\int \frac{dt}{1+t^5}$ .

9 Using a power series expansion, approximate  $\int_0^{1/2} \frac{dx}{1+x^5}$  with error  $< 0.001$ .

# Radius of Convergence

1

Show that the series  $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$  converges for  $|x| < \frac{1}{2}$ .

What happens at the points  $x = \frac{1}{2}$  and  $x = -\frac{1}{2}$ ?

Solution

Radius of convergence is

$$R = \lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \rightarrow \infty} \frac{2^n}{\frac{2^{n+1}}{n+1}} = \lim_{n \rightarrow \infty} \frac{n+1}{2} = \frac{1}{2}.$$

Hence the power series converges for  $|x| < \frac{1}{2}$ .

By the Alternating Series Test

If  $x = -\frac{1}{2}$ , the power series is the alternating series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  which converges.

If  $x = \frac{1}{2}$  the power series in question is the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  which diverges.

# Radius of Convergence

2

Find the radius of convergence of  $\sum_{n=1}^{\infty} \frac{(2n)!x^n}{(n!)^2}$ .

**Solution**

Radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \rightarrow \infty} \frac{\frac{(2n)!}{(n!)^2}}{\frac{(2(n+1))!}{((n+1)!)^2}} = \lim_{n \rightarrow \infty} \frac{((n+1)!)^2}{(n!)^2} \frac{(2n)!}{(2n+2)!} \\ &= \lim_{n \rightarrow \infty} \left( \frac{(n+1)!}{n!} \right)^2 \frac{(2n)!}{(2n+2)!} \\ &= \lim_{n \rightarrow \infty} \left( \frac{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)}{1 \cdot 2 \cdot 3 \cdots n} \right)^2 \frac{1 \cdot 2 \cdot 3 \cdots (2n)}{1 \cdot 2 \cdot 3 \cdots (2n) \cdot (2n+1) \cdot (2n+2)} \\ &= \lim_{n \rightarrow \infty} (n+1)^2 \frac{1}{(2n+2)(2n+1)} = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2 + 5n + 2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4 + \frac{5}{n} + \frac{2}{n^2}} = \frac{1}{4}. \end{aligned}$$

# Converges of the Differentiated Series

3 The radius of convergence of the power series  $\sum_{k=0}^{\infty} a_k x^k$  is  $R$ .

What is the radius of convergence of the power series  $\sum_{k=1}^{\infty} k a_k x^{k-1}$ ?

**Solution** The Radius of Convergence of  $\sum_{k=0}^{\infty} a_k x^k$  is

$R = \lim_{k \rightarrow \infty} \frac{|a_k|}{|a_{k+1}|}$  and that of  $\sum_{k=1}^{\infty} k a_k x^{k-1}$  is

This limit is 1.

$$\lim_{k \rightarrow \infty} \frac{|k a_k|}{|(k+1) a_k|} = \lim_{k \rightarrow \infty} \frac{|a_k|}{|a_{k+1}|} \frac{k}{k+1} = \lim_{k \rightarrow \infty} \frac{|a_k|}{|a_{k+1}|} \lim_{k \rightarrow \infty} \frac{k}{k+1} = R.$$

Next use the fact that the limit of a product is the product of the limits.

This limit is  $R$  by the definition.

# Convergence of the Integrated Series

4 The radius of convergence of the power series  $\sum_{k=0}^{\infty} a_k x^k$  is  $R$ .

What is the radius of convergence of the power series  $\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ ?

**Solution**

The Radius of Convergence of  $\sum_{k=0}^{\infty} a_k x^k$  is

$R = \lim_{k \rightarrow \infty} \frac{|a_k|}{|a_{k+1}|}$  and that of  $\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$  is

This limit is 1.

$$\lim_{k \rightarrow \infty} \frac{\frac{|a_k|}{k+1}}{\frac{|a_{k+1}|}{k+2}} = \lim_{k \rightarrow \infty} \frac{|a_k|}{|a_{k+1}|} \frac{k+2}{k+1} = \lim_{k \rightarrow \infty} \frac{|a_k|}{|a_{k+1}|} \lim_{k \rightarrow \infty} \frac{k+2}{k+1} = R.$$

Next use the fact that the limit of a product is the product of the limits.

This limit is  $R$  by the definition.

# Finding Power Series for Functions

**5** Find a power series representation for the function  $f(x) = \frac{x^2}{1-2x}$

and determine the radius of convergence of the power series.

**Solution**

Substitute  $t = 2x$  to the power series

$$\frac{1}{1-t} = 1 + t + t^2 + \dots = \sum_{k=0}^{\infty} t^k \text{ to get}$$

$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + \dots = \sum_{k=0}^{\infty} 2^k x^k. \quad \text{Multiply by } x^2 \text{ to get}$$

$$\frac{x^2}{1-2x} = x^2 + 2x^3 + 2^2 x^4 + \dots = \sum_{k=0}^{\infty} 2^k x^{k+2}. \quad \text{The Radius of Convergence is}$$

$$R = \lim_{k \rightarrow \infty} \frac{2^k}{2^{k+1}} = \lim_{k \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$$

Observe that the function is undefined for  $x = 1/2$ . Hence it is to be expected that a power series for the function will not converge for  $x = 1/2$ .

# Fibonacci Power Series

6 Find the radius of convergence of the power series

$$\frac{x}{1-x-x^2} = F_0 + F_1x + F_2x^2 + \dots \text{ where } F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

are the Fibonacci numbers,  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$ .

**Solution**

The radius of convergence is

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|F_n|}{|F_{n+1}|} &= \lim_{n \rightarrow \infty} \frac{\frac{\alpha^n - \beta^n}{\alpha - \beta}}{\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}} = \lim_{n \rightarrow \infty} \frac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{\alpha} - \left(\frac{\beta}{\alpha}\right)^n \frac{1}{\alpha}}{1 - \left(\frac{\beta}{\alpha}\right)^{n+1}} \\ &= \frac{1}{\alpha} \end{aligned}$$

Use the fact that  $|\beta| < \alpha$ . Hence  $\lim_{n \rightarrow \infty} \left(\frac{\beta}{\alpha}\right)^n = 0$ .

These terms have the limit 0.

# Finding Power Series

7

Find power series representations for the

$$\text{functions } f(x) = \frac{1}{1-x^2} \text{ and } g(x) = \frac{x^2}{x+1}.$$

**Solution**

Substitute  $t = x^2$  to the power series

$$\frac{1}{1-t} = 1 + t + t^2 + \dots = \sum_{k=0}^{\infty} t^k \text{ to get}$$

$$f(x) = \frac{1}{1-x^2} = 1 + x^2 + (x^2)^2 + \dots = \sum_{k=0}^{\infty} x^{2k}.$$

Substitute  $t = -x$  to the above power series in  $t$  to get

$$\frac{1}{x+1} = \frac{1}{1-(-x)} = 1 - x + x^2 - \dots = \sum_{k=0}^{\infty} (-x)^k. \text{ Multiply by } x^2 \text{ to get}$$

$$g(x) = \frac{x^2}{x+1} = x^2 - x^3 + \dots = \sum_{k=0}^{\infty} (-1)^k x^{k+2}.$$

# Power Series for Indefinite Integrals

8 Find power series representations for the integrals  $\int \frac{dx}{1-x}$  and  $\int \frac{dt}{1+t^5}$ .

**Solution** Integrate the power series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{k=0}^{\infty} x^k \text{ term by term to get}$$

$$\int \frac{dx}{1-x} = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = C + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}. \quad \text{Here } C \text{ is the constant of integration.}$$

Substitute  $x = -t^5$  to the above power series for  $1/(1-x)$  to get

$$\frac{1}{1+t^5} = 1 - t^5 + t^{10} - \dots = \sum_{k=0}^{\infty} (-1)^k t^{5k}. \quad \text{Integrate the power series term by term to get}$$

$$\int \frac{dt}{1+t^5} = C + t - \frac{t^6}{6} + \frac{t^{11}}{11} - \dots = C + \sum_{k=0}^{\infty} (-1)^k \frac{t^{5k+1}}{5k+1}.$$

# Power Series for Definite Integrals

9 Using a power series expansion, approximate  $\int_0^{1/2} \frac{dx}{1+x^5}$

with error  $< 0.001$ .

**Solution** In the previous problem we found out that

$$\frac{1}{1+x^5} = \sum_{k=0}^{\infty} (-1)^k x^{5k}. \quad \text{Integrate this series.}$$

These operations can be swapped.

$$\int_0^{1/2} \frac{dx}{1+x^5} = \int_0^{1/2} \left( \sum_{k=0}^{\infty} (-1)^k x^{5k} \right) dx = \sum_{k=0}^{\infty} \left( \int_0^{1/2} (-1)^k x^{5k} dx \right) = \sum_{k=0}^{\infty} \left( (-1)^k \frac{x^{5k+1}}{5k+1} \Big|_0^{1/2} \right)$$

$$\int_0^{1/2} \frac{dx}{1+x^5} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{5k+1} (5k+1)} \approx \frac{1}{2} - \frac{1}{384} \approx 0.4973$$

This approximation is already accurate enough since the series is alternating and the first term left out is  $1/22528 < 0.001$ .