Solved Problems about Power Series

Summary of Power Series

Power Series $S(x)=\sum_{k=0}^{\infty} a_k x^k$, assume $a_k \neq 0$ for all k.

Basic Power Series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$

Radius of Convergence $R = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}$ (By the Ratio Test) $= \lim_{k\to\infty} \frac{1}{\sqrt[k]{|a_k|}}$ (By the Root Test)

The Power Series $S(x)$ converges if $|x| < R$ and diverges if $|x| > R$.

Differentiation $S'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$

The swapping of summation and integration is valid in the interval of convergence.

Integration $\int S(x)\,dx = \int \left(\sum_{k=0}^{\infty} a_k x^k\right) dx = \sum_{k=0}^{\infty} \left(\int a_k x^k dx\right) = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$

$\int_a^b S(x)\,dx = \int_a^b \left(\sum_{k=0}^{\infty} a_k x^k\right) dx = \sum_{k=0}^{\infty} \left(\int_a^b a_k x^k dx\right) = \sum_{k=0}^{\infty} \left(\frac{a_k}{k+1} x^{k+1}\Big|_a^b\right)$

Overview of Problems

1. Show that the series $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$ converges for $|x| < \frac{1}{2}$.
2. Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(2n)!\,x^n}{(n!)^2}$.

In Problems 3 and 4, the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$ is R.

3. What is the radius of convergence of the power series $\sum_{k=1}^{\infty} k a_k x^{k-1}$?
4. What is the radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$?
5. Find a power series representation for the function $f(x) = \frac{x^2}{1-2x}$ and determine the radius of convergence of the power series.

Overview of Problems

6 Find the radius of convergence of the power series

$$\frac{x}{1-x-x^2} = F_0 + F_1x + F_2x^2 + \cdots \text{ where } F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

are the Fibonacci numbers, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

7 Find power series representations for the functions $f(x) = \frac{1}{1-x^2}$ and $g(x) = \frac{x^2}{x+1}$.

8 Find power series representations for the integrals $\int \frac{dx}{1-x}$ and $\int \frac{dt}{1+t^5}$.

9 Using a power series expansion, approximate $\int_0^{1/2} \frac{dx}{1+x^5}$ with error <0.001.

Radius of Convergence

1 Show that the series $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$ converges for $|x| < \frac{1}{2}$.

What happens at the points $x = \frac{1}{2}$ and $x = -\frac{1}{2}$?

Solution Radius of convergence is

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\frac{2^n}{n}}{\frac{2^{n+1}}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n}\frac{1}{2} = \frac{1}{2}.$$

Hence the power series converges for $|x| < \frac{1}{2}$.

By the Alternating Series Test

If $x = -\frac{1}{2}$, the power series is the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges.

If $x = \frac{1}{2}$ the power series in question is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

Radius of Convergence

2 Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(2n)!\,x^n}{(n!)^2}$.

Solution Radius of convergence is

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\frac{(2n)!}{(n!)^2}}{\frac{(2(n+1))!}{((n+1)!)^2}} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(n!)^2}\frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \left(\frac{1\cdot 2\cdot 3\cdot\cdots\cdot n\cdot(n+1)}{1\cdot 2\cdot 3\cdot\cdots\cdot n}\right)^2 \frac{1\cdot 2\cdot 3\cdot\cdots\cdot(2n)}{1\cdot 2\cdot 3\cdot\cdots\cdot(2n)\cdot(2n+1)\cdot(2n+2)}$$

$$= \lim_{n\to\infty} (n+1)^2 \frac{1}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+5n+2} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{5}{n}+\frac{2}{n^2}} = \frac{1}{4}.$$

Converges of the Differentiated Series

3 The radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$ is R.

What is the radius of convergence of the power series $\sum_{k=1}^{\infty} k a_k x^{k-1}$?

Solution The Radius of Convergence of $\sum_{k=0}^{\infty} a_k x^k$ is

$R = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}$ and that of $\sum_{k=1}^{\infty} k a_k x^{k-1}$ is

$$\lim_{k\to\infty} \frac{|k a_k|}{|(k+1) a_k|} = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} \frac{k}{k+1} = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} \lim_{k\to\infty} \frac{k}{k+1} = R.$$

This limit is 1.

Next use the fact that the limit of a product is the product of the limits.

This limit is R by the definition.

Convergence of the Integrated Series

4 The radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$ is R.

What is the radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$?

Solution The Radius of Convergence of $\sum_{k=0}^{\infty} a_k x^k$ is

$R = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}$ and that of $\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ is

$$\lim_{k\to\infty} \frac{\left|\frac{a_k}{k+1}\right|}{\left|\frac{a_{k+1}}{k+2}\right|} = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} \frac{k+2}{k+1} = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} \lim_{k\to\infty} \frac{k+2}{k+1} = R.$$

This limit is 1.

Next use the fact that the limit of a product is the product of the limits.

This limit is R by the definition.

Finding Power Series for Functions

5 Find a power series representation for the function $f(x) = \frac{x^2}{1-2x}$ and determine the radius of convergence of the power series.

Solution Substitute $t = 2x$ to the power series

$\frac{1}{1-t} = 1 + t + t^2 + \cdots = \sum_{k=0}^{\infty} t^k$ to get

$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + \cdots = \sum_{k=0}^{\infty} 2^k x^k.$ Multiply by x^2 to get

$\frac{x^2}{1-2x} = x^2 + 2x^3 + 2^2 x^4 + \cdots = \sum_{k=0}^{\infty} 2^k x^{k+2}.$ The Radius of Convergence is

$R = \lim_{k\to\infty} \frac{2^k}{2^{k+1}} = \lim_{k\to\infty} \frac{1}{2} = \frac{1}{2}$

Observe that the function is undefined for $x = ½$. Hence it is to be expected that a power series for the function will not converge for $x = ½$.

Fibonacci Power Series

6 Find the radius of convergence of the power series

$$\frac{x}{1-x-x^2} = F_0 + F_1 x + F_2 x^2 + \cdots \text{ where } F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

are the Fibonacci numbers, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Solution The radius of convergence is

$$\lim_{n\to\infty} \frac{|F_n|}{|F_{n+1}|} = \lim_{n\to\infty} \frac{\frac{\alpha^n - \beta^n}{\alpha - \beta}}{\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}} = \lim_{n\to\infty} \frac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{\alpha} - \left(\frac{\beta}{\alpha}\right)^n \frac{1}{\alpha}}{1 - \left(\frac{\beta}{\alpha}\right)^{n+1}}$$

$$= \frac{1}{\alpha}$$

Use the fact that $|\beta| < \alpha$. Hence $\lim_{n\to\infty} \left(\frac{\beta}{\alpha}\right)^n = 0$.

These terms have the limit 0.

Finding Power Series

7 Find power series representations for the functions $f(x) = \frac{1}{1-x^2}$ and $g(x) = \frac{x^2}{x+1}$.

Solution Substitute $t = x^2$ to the power series

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots = \sum_{k=0}^{\infty} t^k \text{ to get}$$

$$f(x) = \frac{1}{1-x^2} = 1 + x^2 + \left(x^2\right)^2 + \cdots = \sum_{k=0}^{\infty} x^{2k}.$$

Substitute $t = -x$ to the above power series in t to get

$$\frac{1}{x+1} = \frac{1}{1-(-x)} = 1 - x + x^2 - \cdots = \sum_{k=0}^{\infty} (-x)^k.$$ Multiply by x^2 to get

$$g(x) = \frac{x^2}{x+1} = x^2 - x^3 + \cdots = \sum_{k=0}^{\infty} (-1)^k x^{k+2}.$$

Power Series for Indefinite Integrals

8 Find power series representations for the integrals $\int \frac{dx}{1-x}$ and $\int \frac{dt}{1+t^5}$.

Solution Integrate the power series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k \text{ term by term to get}$$

$$\int \frac{dx}{1-x} = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = C + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.$$

Here C is the constant of integration.

Substitute $x = -t^5$ to the above power series for $1/(1-x)$ to get

$$\frac{1}{1+t^5} = 1 - t^5 + t^{10} - \cdots = \sum_{k=0}^{\infty} (-1)^k t^{5k}.$$ Integrate the power series term by term to get

$$\int \frac{dt}{1+t^5} = C + t - \frac{t^6}{6} + \frac{t^{11}}{11} - \cdots = C + \sum_{k=0}^{\infty} (-1)^k \frac{t^{5k+1}}{5k+1}.$$

Power Series for Definite Integrals

9 Using a power series expansion, approximate $\int_0^{1/2} \frac{dx}{1+x^5}$ with error <0.001.

Solution In the previous problem we found out that

$\frac{1}{1+x^5} = \sum_{k=0}^{\infty} (-1)^k x^{5k}$. Integrate this series.

These operations can be swapped.

$$\int_0^{1/2} \frac{dx}{1+x^5} = \int_0^{1/2} \left(\sum_{k=0}^{\infty} (-1)^k x^{5k} \right) dx = \sum_{k=0}^{\infty} \left(\int_0^{1/2} (-1)^k x^{5k} dx \right) = \sum_{k=0}^{\infty} \left((-1)^k \frac{x^{5k+1}}{5k+1} \Big|_0^{1/2} \right)$$

$$\int_0^{1/2} \frac{dt}{1+x^5} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{5k+1}(5k+1)} \approx \frac{1}{2} - \frac{1}{384} \approx 0.4973$$

This approximation is already accurate enough since the series is alternating and the first term left out is $1/22528 < 0.001$.